GWE1

We have wave eqn for $\vec{E}, \vec{B}$ or, choosing a suitable gauge, for potentials $\vec{A}, \phi$. To solve wave eqn, let's get its Greens function

Analogy $\quad -\nabla^2 \phi = \rho(r)/\epsilon_0$

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \int dr' \, \rho(r')/|r-r'|$$

$$G(r, r') = \frac{1}{4\pi} \frac{1}{|r-r'|} \quad \text{for } \nabla^2 \text{ operator}$$

So do same for

$$\left.\begin{aligned} \left(\nabla^2 - \tfrac{1}{c^2}\partial^2/\partial t^2\right)\phi &= -4\pi\rho \\ \left(\nabla^2 - \tfrac{1}{c^2}\partial^2/\partial t^2\right)\vec{A} &= -\tfrac{4\pi}{c}\vec{J} \end{aligned}\right\} \text{ Lorenz gauge}$$

$-\rho/\epsilon_0$

$-\mu_0 j$

Fourier transform G with respect to time

$R = r - r'$

$t = t - t'$

$k = \omega/c$

$$G(\vec{R}, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{G}(\vec{R}, \omega)$$

$$(\nabla^2 + k^2)\tilde{G}(\vec{R}, \omega) = -4\pi\delta(\vec{R})$$

if we knew $\nabla^2 - \frac{1}{c^2}\partial^2/\partial t^2 \; G(\vec{r}t\,\vec{r}'t') = -4\pi\delta(\vec{r}-\vec{r}')\delta(t-t')$

then $\quad \phi(rt) = \int d^3r'\,dt' \, G(r-r', t-t') \dfrac{\rho(r't')}{4\pi\epsilon_0}$

GWE1'

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] G(R\tau) = -4\pi\delta(R)\delta(\tau)$$

$$\int \frac{d\omega}{2\pi}\left[\nabla^2 + \frac{\omega^2}{c^2}\right]\tilde{G}(R\omega)\, e^{-i\omega\tau} = -4\pi\,\delta(R)\int \frac{d\omega}{2\pi}\, 1\, e^{-i\omega\tau}$$

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{G}(R\omega) = -4\pi\delta(R)$$

GWE 2

We know how to solve in static case $\omega \to 0$

$$\tilde{G}(R,0) = 1/R \qquad R = |\vec{R}|$$

In fact, we can find soln for G which depends only on $R = |\vec{R}|$ at all ω. The Laplacian simplifies to

$$\left[\frac{1}{R}\frac{d^2}{dR^2}(R\tilde{G}) + k^2 R\tilde{G}\right] = -4\pi\delta(\vec{R})$$

But $R\,\delta(\vec{R}) = 0$. obviously true for $\vec{R} \neq 0$

But even at $\vec{R} = 0$, Basically $x\,\delta(x) = 0$.

$$\frac{d^2}{dR^2}(R\tilde{G}) = -k^2 R\tilde{G}$$

$$R\tilde{G}(R,\omega) = Ae^{ikR} + Be^{-ikR}$$

Must have $A + B = 1$ because $\tilde{G}(R,0) = 1/R$

$$\tilde{G}(R,\omega) = \frac{Ae^{ikR}}{R} + \frac{(1-A)e^{-ikR}}{R}$$

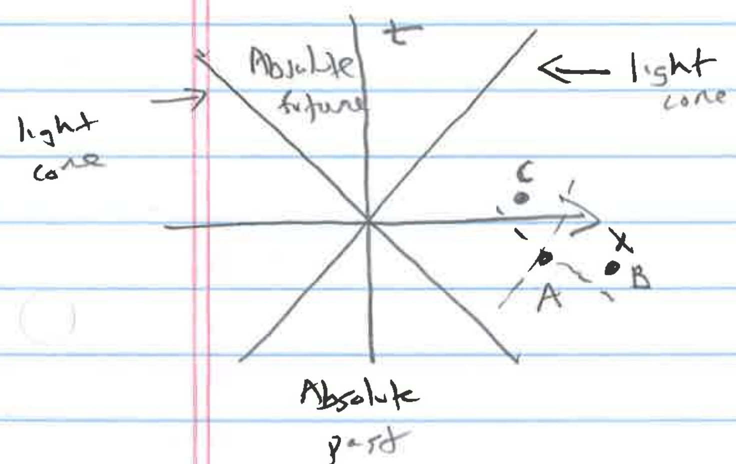
GWE3

Limiting cases: $A=1$, $A=0$

$$\tilde{G}^{\pm}(\vec{R},\omega) = \frac{e^{\pm ikR}}{R} \qquad k=\omega/c$$

$$G^{\pm}(\vec{R},\tau) = \frac{1}{R}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(\tau\mp R/c)} = \frac{1}{R}\delta(\tau\mp R/c)$$

In single spatial dimension



Particle starting at origin can only reach positions in forward light cone

Only particles in backward light cone can reach origin

A + B are space-like separated
no information/particle can get from A to B

A + B can be made simultaneous in an inertial ref. frame moving fast enough

A + C are time like separated
they can never be made to appear simultaneous.

$$G^{+}(\vec{R},\tau) \equiv G_{ret}(\vec{R},\tau) \qquad G^{+}(\vec{R},\tau) = \frac{1}{R}\delta(\tau - R/c)$$

$$G^{-}(\vec{R},\tau) \equiv G_{adv}(\vec{R},\tau)$$

is non zero iff $\tau = R/c$

$t-t' = R/c \quad R = |\vec{r}-\vec{r}'| > 0$

$t > t'$

so G^{+} is physically meaningful one

GWE4

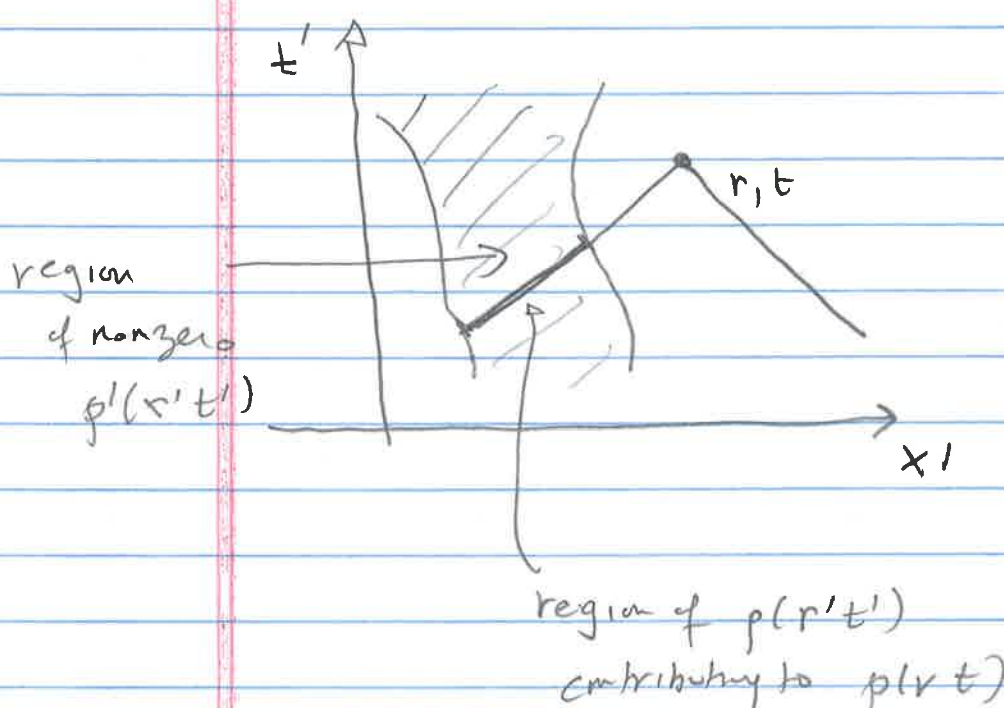
$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(r t) = -4\pi\rho(r t)$$

Sln is

$$\phi(r t) = \phi_0(r, t) + \int d^3r' \int dt' \, \frac{\rho(r' t')}{|r - r'|} \, \delta\left(t' - (t - R/c)\right)$$

any soln to homogeneous wave eqn

Usually in EM we know conditions of far from system at all times and ϕ_0 is potential of incident radiation while integral term is radiation emitted

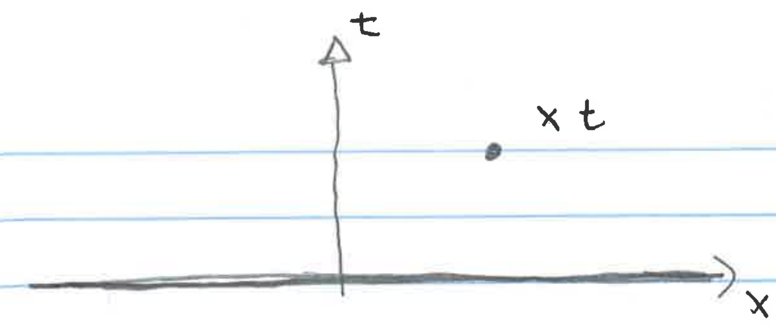


δ function non zero only on backwards light cone

$$t' = t - R/c$$

Contrast with Diffusion Eqn $\quad \phi(x t) = \int dx' \, \phi(x' 0) \, \frac{e^{-\frac{(x-x')^2}{4Dt}}}{\sqrt{4\pi D t}}$

GWE5



values of $\phi(x' 0)$ for any x' can influence value at x!

Typical diffusion constant:

H_2O in air $\quad D = 0.219\ cm^2/s$

Negligible for $\quad \dfrac{(x-x')^2}{4Dt} \gtrsim 100 \qquad e^{-100}$

$$(x-x')^2 \gtrsim 87.6\, t \quad (cm^2)$$

$t = 10 sec \qquad (x-x') \gtrsim 30\ cm = 0.3\ m$

so velocity of $\quad 0.03\ m/s = 10^{-10}\ c.$

speed of light

GWE 6

Solve for field of uniformly moving charge $(\vec{V} = v\hat{x})$

$$\rho(\vec{r}t) = q\,\delta(\vec{r} - vt\hat{x})$$

$$J(\vec{r}t) = qv\hat{x}\,\delta(\vec{r} - vt\hat{x})$$

Abbreviate $\quad x_t = x - vt \qquad \gamma = (1 - v^2/c^2)^{-1/2}$

ϕ can only depend on x, t in combination $x - vt$

$$\phi(\vec{r}t) = \int \frac{d^3k}{(2\pi)^3}\,\phi_k\, e^{i\vec{k}\cdot\vec{r}}\, e^{-ik_x vt}$$

$$(\nabla^2 - 1/c^2\, \partial^2/\partial t^2)\,\phi(\vec{r}t) = -4\pi\rho(\vec{r}t) \qquad -\rho/\epsilon_0$$

$$\left(k^2 - \frac{v^2}{c^2}k_x^2\right)\phi_k = 4\pi q \qquad q/\epsilon_0$$

Since $\quad \rho(\vec{r}t) = q\,\delta(\vec{r} - vt\hat{x}) = q\int \frac{d^3k}{(2\pi)^3}\, e^{ik\cdot(r - vt\hat{x})}$

we can find ϕ of same form, ie

Thus $\quad \phi(\vec{r}t) = \int \frac{d^3k}{(2\pi)^3}\, \frac{4\pi q}{\gamma^{-2}k_x^2 + k_y^2 + k_z^2}\, e^{i(k_x x_t + k_y y + k_z z)} \qquad q/\epsilon_0$

Defining $\quad k_x' = k_x/\gamma \qquad k_y' = k_y \quad k_z' = k_z$

$$\phi(\vec{r}t) = \gamma \int \frac{d^3k'}{(2\pi)^3}\, \frac{4\pi q}{k'^2}\, e^{i(\gamma k_x' x_t + k_y' y + k_z' z)} \qquad q/\epsilon_0$$

But FT of $4\pi/k^2$ is $1/r$

$$= \gamma \frac{q}{\sqrt{\gamma^2(x - vt)^2 + y^2 + z^2}} \qquad q/4\pi\epsilon_0$$

GWK 7

Can get $\vec{A}$ trivially $A_y = A_z = 0$

since $J_x = J_y = 0$. Eqn for A_x mimics ϕ

$$\vec{A}(\vec{r}t) = \frac{\vec{v}}{c}\phi = \gamma \frac{q}{c\left(\gamma^2(x-vt)^2+y^2+z^2\right)^{1/2}}\vec{v}$$

$\phi, \vec{A}$ depend only on t via $x-vt$ so

$$\partial/\partial t \rightarrow -\vec{v}\cdot\nabla$$

$$\vec{E} = -\vec{\nabla}\phi - 1/c\, \partial\vec{A}/\partial t = -\nabla\phi + 1/c^2\, \vec{v}(\vec{v}\cdot\vec{\nabla}\phi)$$

$$= q\left(1-v^2/c^2\right)\frac{\vec{R}}{\left[(x-vt)^2+(1-\frac{v^2}{c^2})(y^2+z^2)\right]^{3/2}}$$

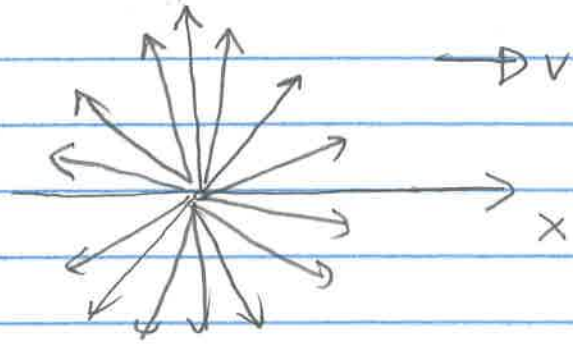
$$\vec{R} = \vec{r} - \vec{r}_0(t)$$

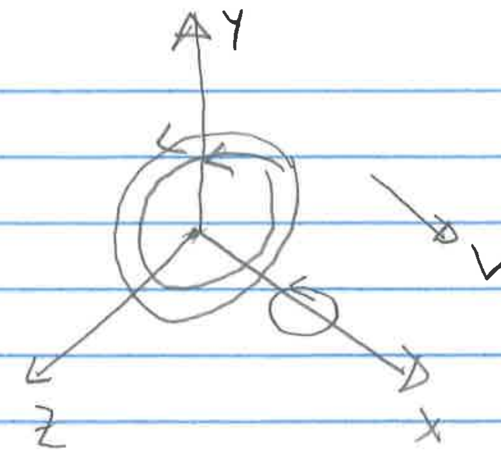
$$\vec{r}_0(t) = (vt, 0, 0)$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{-\vec{v}}{c}\times\vec{\nabla}\phi = \frac{1}{c}\vec{v}\times\vec{E}$$

GWE8

Pictures





(1) $\vec{E}$ and $\vec{B}$ can be expressed in term of actual instantaneous position of charge. That is

$$\vec{E}(\underbrace{x,y,z}_{\text{location at which } \vec{E} \text{ is evaluated}},\ t) \quad \text{depends on} \quad \begin{matrix} x-vt \\ y \\ z \end{matrix} = \vec{r} - \vec{r}_0(t)$$

time

(2) $\underbrace{\vec{R} \perp \vec{v}}_{x-vt=0}$ $\quad \vec{E} = \frac{q\vec{R}}{R^3} \frac{1}{\sqrt{1-v^2/c^2}}$ } enhanced

$\vec{R} \parallel \vec{v}$ $\quad \vec{E} = \frac{q\vec{R}}{R^3}\left(1-v^2/c^2\right)$ } suppressed

$y=z=0$

(3) $\vec{E} \parallel \vec{R}$ $\quad \vec{S} \propto \vec{E} \times \vec{B}$ $\quad$ so $\vec{S} \perp \vec{R}$

Poynting vector

no radial outflow of energy

Accelerated charges
much of this will change

(1) Need retarded position

(2)
(3) } Part of fields due to acceleration are focussed forward (synchrotron uses this) and power is radiated.