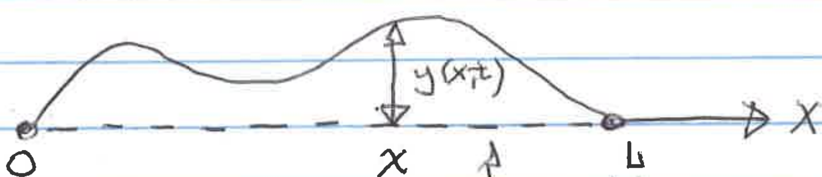
S1

We will spend much time this quarter on EM waves. Let's begin by reviewing in classical context, and more simple (1D).

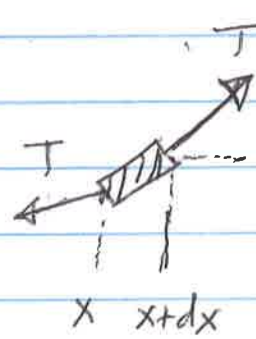
Wave eqn comes immediately from Maxwell eqns. Let's derive it for vibrating string



equilibrium position $y(x,t)=0$

$y(x,t)$ = displacement of string at position x and time t

$\boxed{Q}$ Consider $F = ma$ for a small piece of string



Assume Tension T is same everywhere in string

$$F_y = T\frac{\partial y}{\partial x}\bigg|_{x+dx} - T\frac{\partial y}{\partial x}\bigg|_{x}$$

$$\approx T\,dx\,\frac{\partial^2 y}{\partial x^2}$$

$$\overbrace{T\,dx\,\frac{\partial^2 y}{\partial x^2}}^{F_y} = \overbrace{\mu\,dx}^{m}\,\overbrace{\frac{\partial^2 y}{\partial t^2}}^{a_y}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} = v^2\frac{\partial^2 y}{\partial x^2}$$

with $v \equiv \sqrt{T/\mu}$ units $\left(\frac{ML/T^2}{M/L}\right)^{1/2} = L/T$

$\boxed{Q}$ Separation of variables $y(x,t) = f(x)g(t)$

$$f(x)g''(t) = v^2 f''(x)g(t)$$

Must equal constant $g''(t)/g(t) = v^2 f''(x)/f(x) \equiv -\omega^2$

$$g(t) = A\sin\omega t + B\cos\omega t$$

$$f(x) = C\sin kx + D\cos kx \qquad k = \omega/v$$

$\omega = vk$

Just as in EM boundary conditions are crucial

$$y(x=0,t) = y(x=L,t) = 0 \Rightarrow D = 0 \qquad \boxed{Q}$$

$$k = \pi n/L$$

$$y(x,t) = \sum_n \sin\frac{n\pi x}{L}\left\{a_n\sin\frac{n\pi vt}{L} + b_n\cos\frac{n\pi vt}{L}\right\}$$

$\boxed{Q}$ why? Wave eqn is linear.

S3

$\boxed{Q}$ Next we impose $t=0$ boundary conditions

aka "initial conditions"

We must be given $y(x,t=0)$ $\partial y/\partial t\,(x,t=0)$

initial shape & velocity of string

$$y(x,t=0) = \sum_n \sin\frac{n\pi x}{L}\, b_n$$

$$\partial y/\partial t\,(x,t=0) = \sum_n \sin\frac{n\pi x}{L}\,\frac{n\pi v}{L}\,a_n$$

$\boxed{Q}$ Invert these Fourier series

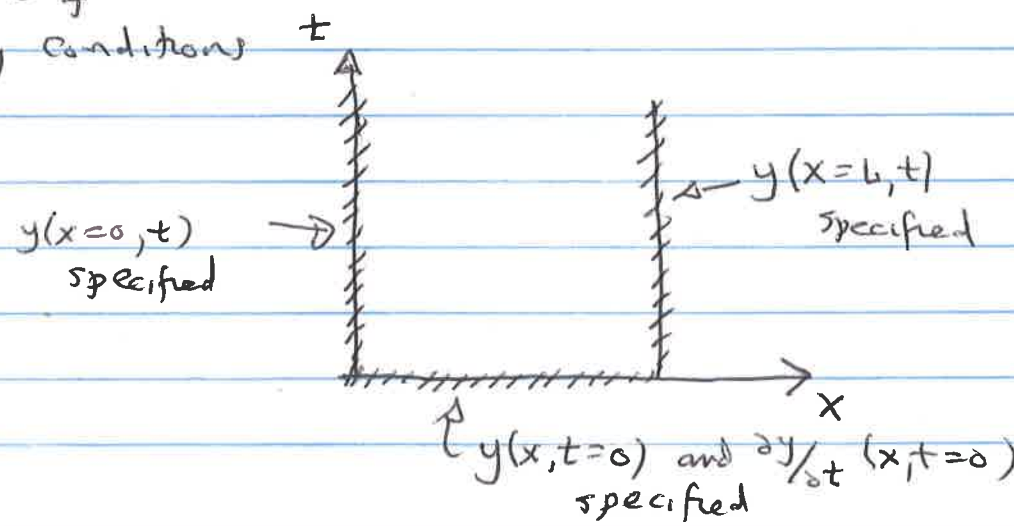
$$b_m = \frac{2}{L}\int_0^L dx\, y(x,t=0)\sin\frac{m\pi x}{L}$$

$$a_m = \frac{L}{n\pi v}\,\frac{2}{L}\int_0^L dx\,\frac{\partial y}{\partial t}(x,t=0)\sin\frac{m\pi x}{L}$$

Using orthogonality $\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \frac{L}{2}\delta_{nm}$

$\boxed{Q}$

Picture of Boundary conditions



$\boxed{Q}$

Green's functions from plugging a_m, b_m in

For simplicity, assume $\partial y/\partial t = 0$

$$y(x,t) = \sum_n \sin\frac{n\pi x}{L} \int_0^L \frac{2}{L}\, dx'\, y(x', t=0) \sin\frac{m\pi x'}{L}$$

$$= \int_0^L dx' \left\{ \sum_n \frac{2}{L} \sin\frac{n\pi x}{L} \sin\frac{n\pi x'}{L} \sin\frac{n\pi v t}{L} \right\} y(x', 0)$$

$$\equiv G(x, x'; t)$$

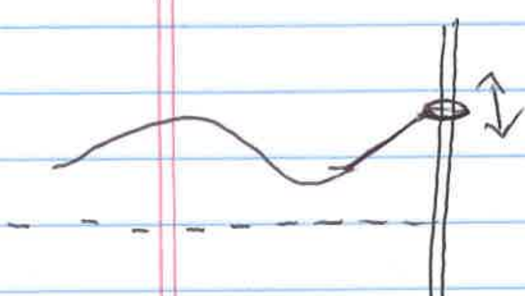
$\boxed{Q}$ another name for Green's function? "Propagator". why?

In EM have different boundary conditions.

Perhaps potential V is specified on boundary, or

maybe you know $\vec{E} = -\vec{\nabla} V$ is $\perp$ to surface of metal

$\boxed{Q}$ → Analog for waves on string?



terminate string in ring of mass m

assume slides w/o friction

$x = L$

in y direction

$$\cancel{\partial y/\partial x \big|_{x=L}} \quad T \frac{\partial y}{\partial x}(x=L, t) = m_{ring}\, a_{ring}$$

Take limit $m_{ring} \to 0$ $\qquad \partial y/\partial x\,(x=L, t) = 0$

rather than $y(x=L, t) = 0$

Greens functions : ← disorder

$G(x, x', t)$ vs $G(x - x', t)$

Disturbance at x' propagates to x in time t

Partial Diff Eqn → field theory

create particle at x and ask for probability amplitude for finding it at x' at time t later

$$\to \quad -i \langle 0 | \psi(x, t) \, \psi^\dagger(x', 0) | 0 \rangle$$

↑ ground state

Boundary conditions → different types of G : retarded, advanced, time ordered

Defined (Expt) ↗ retarded

Easy to compute (Feynman diagrams) ↗ time ordered

⟷ Kramers-Kronig relations

In field theory what happens

$e^- \xrightarrow{k, \omega}$ moving hopping along

emits photon/phonon rest mass renormalized

polarons

In EM $B(x', 0) \to E(x' + dx, dt) \to B(x' + \ldots, \ldots)$

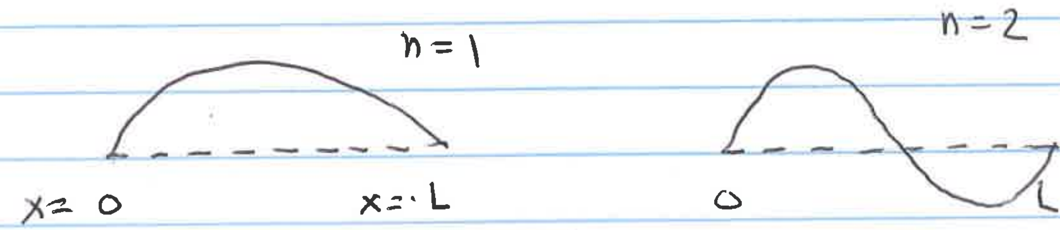
time varying B

by Faraday

by ?

S5.

When discussing vibrating strings we often think of the individual modes:



[Q] why? Music perhaps but also these are special "normal modes" which do not mix.

Fourier series can be used to build up other shapes.

Indeed $y(x,t) = e^{-(x-vt)^2/\xi^2}$

is a solution of wave eqn.

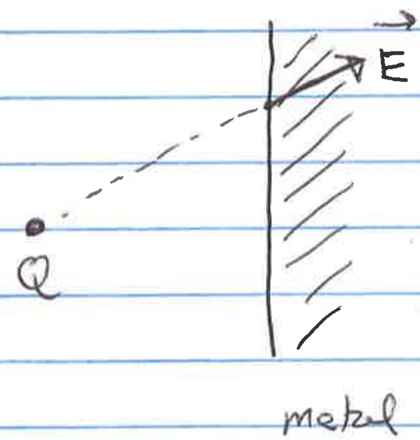
[Q] Can it be represented using our sin, cos?
No, because will not satisfy boundary conditions.

propagates without changing its shape

[Q] Does this happen in QM? Why or why not? How are wave and Schroedinger Eqns different?

56.

Again analogs to EM in dealing with boundary conditions

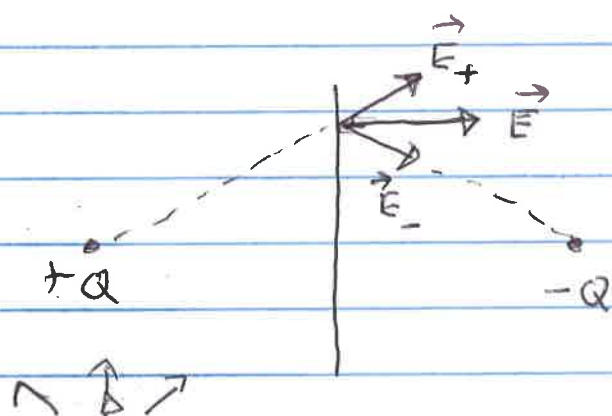


$\boxed{Q}$ How to solve?

What needs to happen?

$E_y = 0$ at surface

place "image charge"

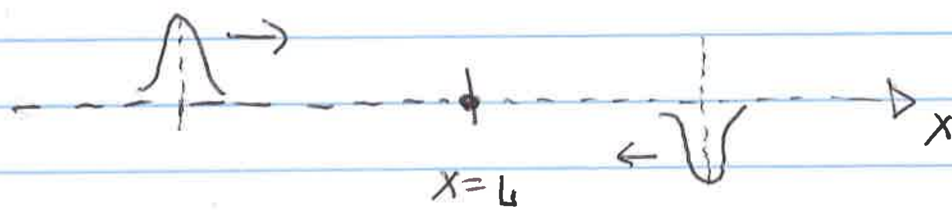


Still obey $\nabla^2 V = 0$ here

$\nabla \cdot E = \rho/\epsilon_0$ here } since have not change ρ any in this region

But have fixed up boundary condition.

$\boxed{R}$ String analog

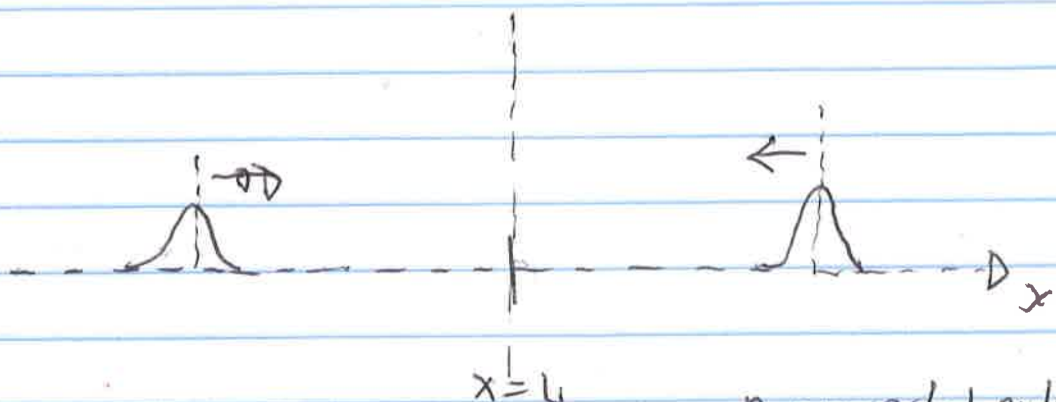


Imaginary inverted pulse

sum of pulses will always have

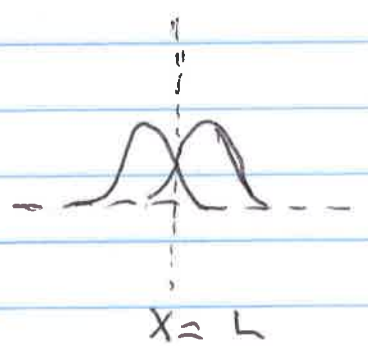
$y(x = L, t) = 0$

$\boxed{Q}$ What about the ring termination?



Noninverted pulse will always have

$\partial y / \partial x \, (x = L, t) = 0$



The two $\partial y / \partial x$ values are equal in magnitude and opposite in sign.

Summary of all this:

Wave eqn for vibrating string (1D)

- follows from $F=ma$
- solution depends on boundary conditions (along edges in x,t plane)
- can get Green function
- analogs of specifying V or $E \perp$ surface
- analogs of "method of images"

We will also spend time this quarter ← and last quarter?

discussing $\vec{E}, \vec{B}$ in physical media.

$\vec{E}$ field polarizes charge in object → additional $\vec{E}$ field

Likewise $\vec{B}$ field can magnetize object → additional $\vec{B}$ field

an especially rich topic in CM physics

additional $\vec{B}$ in same direction as original $\vec{B}$: paramagnetic

opposing original $\vec{B}$: diamagnetic

$\vec{B}$ arises spontaneously w/o "external" B : ferromagnetic

$\boxed{Q}$

linear response

$$M = \chi B$$

$$M(q\,w) = \chi(q\,w)\, B(q, w)$$

susceptibility, coefficient relating B to M, can depend on $\vec{q}$, w of $\vec{B}$

$\boxed{Q}$ Is there an analog of this for vibrating string, eg we could make $\mu \to \mu(x)$ so mass of string depends on position. Would this be like polarization, magnetization?

D1

$f(x \pm vt)$ obeys the wave eqn for an arbitrary function f

This is pretty amazing (we often think of solving differential eqn as resulting in certain specific functions)

Physically there are solns of wave eqn which travel without changing their shape $f(\xi)$

Q why does this happen? Is it true for other pdes like Schrodinger Eqn or diffusion Eqn? How do they differ in structure from wave eqn.

Q 1D Diffusion

$$\frac{\partial y(x,t)}{\partial t} = D \frac{\partial^2 y(x,t)}{\partial x^2}$$

instead of $\partial^2 y/\partial t^2$

x and t appear in fundamentally different way. space and time are not equally treated by diffusion Eqn.

Q Schroedinger Eqn?

How can you make space/time appear more symmetrically in QM? Answer is Dirac Eqn (or Klein-Gordon Eqn)

Schroedinger eqn is a limit of Dirac Eqn.

D2

Q How to solve diffusion eqn?

Same idea of separation of variables $y(x,t) = f(x)g(t)$

$$f(x)g'(t) = D f''(x) g(t)$$

$$g'(t)/g(t) = D f''(x)/f(x) = -Dk^2$$

separation constant

$$g(t) = e^{-Dk^2 t}$$

$$f(x) = e^{ikx}$$ ← we will solve

diffusion eqn on full x axis $-\infty < x < \infty$ so k is not quantized to $n\pi/L$

$$y(x,t) = \int_{-\infty}^{\infty} dk\, a(k) e^{ikx} e^{-Dk^2 t}$$

again, Diffusion Eqn is linear

$$y(x,0) = \int_{-\infty}^{\infty} dk\, a(k) e^{ikx}$$

Q
$$\int_{-\infty}^{\infty} \frac{dx}{2\pi} y(x,0) e^{-ikx} = a(k)$$ Fourier integrals

Q Need only to specify $y(x,0)$, not $\frac{\partial y}{\partial t}(x,0)$ as in wave eqn, why? Diffusion eqn is first order in time.

D3

Can get Green's function

$$y(x,t) = \int dk \int \frac{dx'}{2\pi} e^{-ikx'} y(x',0) e^{ikx} e^{-Dk^2t}$$

$$= \int dx' \left\{ \int \frac{dk}{2\pi} e^{ik(x-x')} e^{-Dk^2t} \right\} y(x',0)$$

$G(x,x',t)$ for diffusion eqn.

$\boxed{Q}$: Why does dependence on $x-x'$ jump out for us here but not for wave eqn?
(We broke translation invariance when we pinned string down at $x=0, L$)

Can actually do integral over k in this case.

$\boxed{Q}$ Complete the square

$$-Dtk^2 + ik(x-x') = -Dt\left(k - \frac{i(x-x')}{2Dt}\right)^2 - Dt\frac{(x-x')^2}{4D^2t^2}$$

$\boxed{Q}$ k integration $\int_{-\infty}^{\infty} e^{-a(k-k_0)^2} dk = \sqrt{\frac{\pi}{a}}$

obvious for real k_0. Here k_0 is imaginary. Why does it still work?

D4

For diffusion Eqn

$$G(x-x',t) = \frac{1}{2\pi}\sqrt{\frac{\pi}{Dt}}\, e^{-(x-x')^2/4Dt}$$

$$y(x,t) = \int_{-\infty}^{\infty} dx'\, G(x-x',t)\, y(x',0)$$

If $y(x',0) = \delta(x')$ ← super concentrated density or hot spot

$$y(x,t) = G(x,t) = \frac{1}{\sqrt{4\pi Dt}}\, e^{-(x-x')^2/4Dt}$$

The delta function spreads. It has width

$$\Delta x \sim \sqrt{4Dt}$$

super impt fact about diffusion. width grows as square root of time.

All sorts of applications.

Eg toss fair coin N times $\langle \#\text{heads} \rangle = N/2$

and expected deviation from $N/2$ $\sim \sqrt{N}$

~~N~~ $\boxed{Q}$ If I told you $N = 10^6$ and got #heads = ~~[illegible]~~ 520000 ← unhappy

502,000 ← okay!

expect $\sqrt{N} \sim 10^3$ deviation