E1

Energy in Electric Field

$$U = \sum_{\text{pairs}} q_a q_b / 4\pi\epsilon_0 r_{ab}$$

$$\phi(r_a) = \sum_{b \neq a} q_b / r_{ab}\, 4\pi\epsilon_0$$

$$\rightarrow \quad U = \frac{1}{2}\sum_a q_a \phi(r_a)$$

$$\Rightarrow \tfrac{1}{2}\int \rho(r)\phi(r)\,dr$$

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad E = -\nabla\phi$$

$$-\nabla^2\phi = \rho/\epsilon_0$$

$$U = -\tfrac{1}{2}\epsilon_0 \int \phi\nabla^2\phi\,dr$$

$$= -\tfrac{1}{2}\epsilon_0 \underbrace{\int_S \underset{1/r}{\phi}\, \underset{1/r^2}{\vec{\nabla}\phi}\cdot\hat{n}\,\underset{r^2}{dS}}_{\text{vanish at } r\to\infty} + \tfrac{1}{2}\epsilon_0\int(\nabla\phi)^2\,dr$$

$$\boxed{U = \frac{1}{2}\epsilon_0\int E^2\,dr}$$

$$\nabla\cdot(\phi\nabla\phi)$$

$$= \nabla\phi\cdot\nabla\phi + \phi\nabla^2\phi$$

$$\Rightarrow \int\phi\nabla^2\phi$$

$$= \underbrace{\int\nabla\cdot(\phi\nabla\phi)}_{} - \int|\nabla\phi|^2$$

$$= \int_S \phi\vec{\nabla}\phi\cdot\hat{n}$$

$$\frac{\partial}{\partial x_i}\left(\phi\frac{\partial}{\partial x_i}\phi\right)$$

$$= \frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_i} + \phi\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\phi$$

$$= \nabla\phi\cdot\nabla\phi + \phi\nabla^2\phi$$

E2'

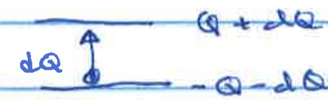
"Physics 9C" argument is based on

capacitors and inductors

As you add charge to a system voltage ~ charge added

$$V = Q/C \qquad C \equiv Q/V$$

$$dU = V\,dQ = Q\,dQ/C$$



$$U = \frac{1}{2C} Q^2$$

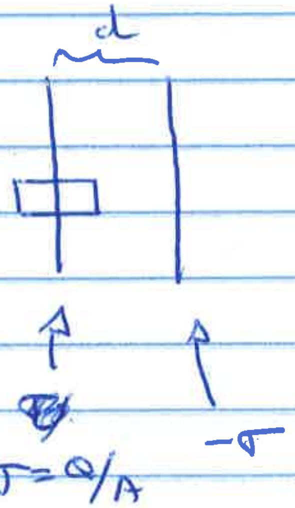
work done by you

But by gauss' law $E = \sigma/\epsilon_0 = Q/A\epsilon_0$

and $C = Q/V = Q/Ed = \epsilon_0 A/d$

so $U = \frac{1}{2}\frac{d}{\epsilon_0 A} A^2 \epsilon_0^2 E^2 = \frac{1}{2}\epsilon_0 (Ad) E^2$

$$U/(Ad) = \tfrac{1}{2}\epsilon_0 E^2$$



$2EA = \sigma A/\epsilon_0$

from each sheet

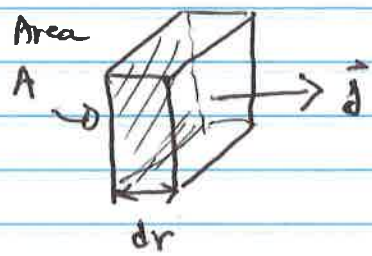
$\boxed{\text{Energy in magnetic field}}$

At first, appears counterintuitive since $\vec{B}$ does no work

$\vec{F} = q\,\vec{v} \times \vec{B}$ is perpendicular to $d\vec{r}$ so $W = \vec{F} \cdot d\vec{r} = 0$.

Crucial observation is Faraday's Law. As you build up a current, $\vec{B}$ will change. This induces $\vec{E}$ which does work.

Quantitatively, consider box with charge Q inside, set it in motion to create current density $\vec{j}$ in short time interval dt



$\vec{j} \parallel d\vec{r}$ direction of motion

$$|\vec{j}| = \frac{Q}{dt}\frac{1}{A}$$

$$dW = -\vec{F} \cdot d\vec{r}$$

$$= -Q\vec{E} \cdot d\vec{r}$$

$$= -|\vec{j}|\,A\,dt\;\vec{E} \cdot d\vec{r}$$

PE is work you do fighting $\vec{E}$ (like gravity)

But since $\vec{j} \parallel d\vec{r}$ we have

$$|\vec{j}|\,d\vec{r} = |dr|\,\vec{j}$$

$$\frac{dW}{dt} = -\underbrace{A\,dr}_{\text{Volume}}\;\vec{j} \cdot \vec{E}$$

Finally $\dfrac{dW}{dt} = -\int \vec{j} \cdot \vec{E}\, d^3r$

Using Amperes law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$

$$\frac{dW}{dt} = \frac{-1}{\mu_0} \int (\vec{\nabla} \times \vec{B}) \cdot \vec{E}\, d^3r$$

integrate by parts (ignore surface term if $\vec{E}, \vec{B}$ vanish at ∞)

$\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = (\vec{\nabla} \times \vec{E}) \cdot \vec{B} - (\vec{\nabla} \times \vec{B}) \cdot \vec{E}$

$$= \frac{-1}{\mu_0} \int \vec{B} \cdot (\vec{\nabla} \times \vec{E})\, d^3r$$

Faraday

$$= +\frac{1}{\mu_0} \int \vec{B} \cdot d\vec{B}/dt\, d^3r$$

$$= \frac{1}{2\mu_0} \frac{d}{dt} \int \vec{B} \cdot \vec{B}\, d^3r$$

Energy density $= \dfrac{|B|^2}{2\mu_0}$

$$\Rightarrow \quad W = \frac{1}{2\mu_0} \int |B|^2\, d^3r$$

E2"

The "lower division" argument is also based on Faraday's law. First, analagous to capacitance, define inductance L as proportionality constant between current I and magnetic flux

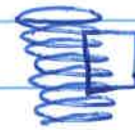
$$\phi_B = LI$$

Then $V = \frac{d\phi_B}{dt} = L\, dI/dt$

$$dU = V\,dQ = L\frac{dI}{dt}\,dQ = L\,dI\,I$$

$$U = \tfrac{1}{2} L I^2$$

$$\oint B\cdot d\ell = \mu_0 I$$



$$B\ell = \mu_0 n \ell I$$

$$B = \mu_0 n I$$

turns / length

$$L = \Phi_B / I$$

$$= (n\ell) BA / I = \mu_0 n^2 \underbrace{\ell A}_{\text{"geometry"}}$$

turns

$\mu_0 n I$

$$U = \frac{1}{2} L I^2 = \frac{1}{2}(\mu_0 n^2 \ell)\left(\frac{B}{\mu_0 n}\right)^2 A$$

$$= \frac{1}{2\mu_0} B^2 \ell A$$

$$\boxed{\frac{U}{\ell A} = \frac{1}{2\mu_0} B^2}$$

E4

Poynting Vector

Time rate of change of energy density is

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right\}$$

$$= \epsilon_0 \vec{E}\cdot\partial\vec{E}/\partial t + \frac{1}{\mu_0}\vec{B}\cdot\partial\vec{B}/\partial t$$

Maxwell Eqns

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

A1 will turn out to be $\partial/\partial t\,(u_E)$

$$\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$$

Q2: How can combine $E\cdot(\nabla\times B)$ and $-B\cdot(\nabla\times E)$?

So $\partial/\partial t\left\{\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right\}$

$$= \epsilon_0\vec{E}\cdot\frac{1}{\epsilon_0}\left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \vec{j}\right) - \frac{1}{\mu_0}\vec{B}\cdot(\vec{\nabla}\times\vec{E})$$

Q1 what is this?

$$= -\vec{j}\cdot\vec{E} - 1/\mu_0\,\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

Q2 Note $\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \frac{\partial}{\partial x_i}\epsilon_{ijk}E_jB_k$

$$= \epsilon_{ijk}\left(\frac{\partial}{\partial x_i}E_j\right)B_k + \epsilon_{ijk}E_j\frac{\partial}{\partial x_i}B_k$$

$$= \epsilon_{kij}\left(\frac{\partial}{\partial x_i}E_j\right)B_k - \epsilon_{jik}\frac{\partial}{\partial x_i}B_kE_j$$

$$= (\nabla\times E)_kB_k - (\nabla\times B)_jE_j$$

$$= \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$$

E4

Consider current as collection of charges

$$\vec{J} = \sum_a q_a \vec{v}_a \delta(\vec{r} - \vec{r}_a)$$

Integrate over volume containing all particles

$$\frac{d}{dt} \int \left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right) dv$$

$$= - \sum_a q_a \vec{v}_a \cdot \vec{E}(r_a) - \frac{1}{\mu_0} \int \vec{\nabla} \cdot (\vec{E} \times \vec{B})\, dv$$

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$- \frac{1}{\mu_0} \int (\vec{E} \times \vec{B}) \cdot \hat{n}\, dS$$

$$q \vec{v} \cdot \vec{E} = q\, \vec{v} \cdot \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\frac{p^2}{2m} \right)$$

So first term is KE!

obvious if nonrelativistic $\vec{p} = m\vec{v}$

and still true if relativistic

$$\vec{p} = m\vec{v} / \sqrt{1 - v^2/c^2}$$

$$\vec{v} = c\vec{p} / \sqrt{p^2 + m^2 c^4}$$

$$\mathcal{E} = c^2 p^2 + m^2 c^4$$

$$\vec{v} \cdot d\vec{p}/dt = \vec{p}c / \sqrt{p^2 + m^2 c^4} \cdot d\vec{p}/dt$$

$$= c \frac{d}{dt} \sqrt{p^2 + m^2 c^2}$$

$$= d\mathcal{E}/dt$$

E5

$$\frac{d}{dt}\left\{ \int \left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2\right) dv + KE \right\}$$

$$= -\frac{1}{\mu_0}\int (\vec{E}\times\vec{B})\cdot\hat{n}\, dA$$

$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ is Poynting vector

Conservation law

time rate of change of energy in volume

field energy
+ kinetic energy

= integral of S over area A bounding volume

energy flux
density

Signs time rate of change of energy
inside volume = flow of energy in
= − flow of energy out.