PHYSICS 200C, SPRING 2017
ELECTRICITY AND MAGNETISM

Assignment Three, Due Friday, April 28, 5:00 pm.

[1.] Show the fields

$$\vec{\mathbf{E}} = E_0 \cos\omega t \cos kx \cos ky \, \hat{\mathbf{z}}$$
$$\vec{\mathbf{B}} = B_0 \sin\omega t \Big(\cos kx \sin ky \, \hat{\mathbf{x}} - \sin kx \cos ky \, \hat{\mathbf{y}} \Big)$$

satisfy the fundamental laws of electricity and magnetism in vacuum assuming $E_0 = \alpha B_0$ and $\omega = \beta c k$ where α, β are certain constants. Find these constants. These fields can exist inside a metal box of dimension π/k in the x and y directions, and arbitrary "height" in the z direction. Sketch what $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ look like in a cross section of the box perpendicular to $\hat{\mathbf{z}}$.

[2.] In class we treated a metal as a cloud of *noninteracting* electrons. We commented that this seemingly outrageous viewpoint is not so bizarre if we understand that the Pauli principle causes the electron kinetic energy to be really huge. Take the equation from class relating the number of electrons N to the largest occupied momentum magnitude $k_{\rm F}$

$$N = 2 \sum_{\vec{k}}^{|\vec{k}|<k_{\rm F}} = 2 \frac{V}{(2\pi)^3} \int d^3k = 2 \frac{V}{(2\pi)^3} \frac{4}{3} \pi k_{\rm F}^3$$

Look up a typical value for the density N/V of electrons in a metal. Use this value to compute $k_{\rm F}$ and then compare the kinetic energy $\hbar^2 k_{\rm F}^2 / 2m_e$ with the interaction energy $e^2/(4\pi\epsilon_0 b)$ where a typical electron separation b is given by $nb^3 = 1$.

Comment 1: Better than looking the density of electrons up would be to estimate it from 'basic chemistry': (i) A typical solid has a (mass) density of a few grams per cm^3; (ii) A typical metal has a few electrons in its outermost shell to form the free electron cloud, the remainder are bound to the nucleus; (iii) A mole has $N_A = 6 \times 10^{23}$ atoms and the mass of each is the number of protons plus the number of neutrons (six for lithium, for example) times their mass $m = 1.67 \times 10^{-24}$ g. (Is it a coincidence that $N_A \times m_p = 1$?)

Comment 2: A famous problem in astrophysics is based on the exact same considerations: Take a typical mass of a neutron star (a few solar masses) and volume. (A neutron star has a radius smaller than Davis, $R \approx 10$ km!) Compute k_F and also the energy E. From $P = -dE/dV$ get the 'degeneracy pressure' and compare it to the pressure due to gravity. They will be roughly equal. Indeed, the high degeneracy pressure is what keeps the neutron star from collapsing.

Show the fields

$$\vec{\mathbf{E}} = E_0 \cos\omega t \cos kx \cos ky\, \hat{\mathbf{z}}$$
$$\vec{\mathbf{B}} = B_0 \sin\omega t \Big(\cos kx \sin ky\, \hat{\mathbf{x}} - \sin kx \cos ky\, \hat{\mathbf{y}}\Big)$$

satisfy the fundamental laws of electricity and magnetism in vacuum assuming $E_0 = \alpha B_0$ and $\omega = \beta ck$ where α, β are certain constants. Find these constants. These fields can exist inside a metal box of dimension π/k in the x and y directions, and arbitrary "height" in the z direction. Sketch what $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ look like in a cross section of the box perpendicular to $\hat{\mathbf{z}}$.

Solution

First note that we need the tangential parts of $\vec{\mathbf{E}}$ to vanish at the surface of the metal, so that the box in question must have $-\pi/2k < x, y < +\pi/2k$ (and not, for example, $0 < x, y < pi/k$).

Check the Maxwell equations involving the divergence:

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\partial E_z}{\partial z} = 0. \tag{1}$$

where we of course used the facts that $E_x = E_y = 0$ and E_z has no z dependence. Similarly,

$$\vec{\nabla} \cdot \vec{\mathbf{B}} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = B_0 \sin\omega t \Big(- k \sin kx \sin ky + k \sin kx \sin ky\Big) = 0 \tag{2}$$

This time we used $B_z = 0$.

Next check the curl equations:

$$\vec{\nabla} \times \vec{\mathbf{E}} = \hat{\mathbf{x}}\frac{\partial E_z}{\partial y} - \hat{\mathbf{y}}\frac{\partial E_z}{\partial x} = E_0 \cos\omega t \Big(- k\, \hat{\mathbf{x}} \cos kx \sin ky + k\, \hat{\mathbf{y}} \sin kx \cos ky\Big)$$
$$\frac{\partial \vec{\mathbf{B}}}{\partial t} = B_0\, \omega \cos\omega t \Big(+ \hat{\mathbf{x}} \cos kx \sin ky - \hat{\mathbf{y}} \sin kx \cos ky\Big) \tag{3}$$

So the condition $\vec{\nabla} \times \vec{\mathbf{E}} = -(1/c)\, \partial\vec{\mathbf{B}}/\partial t$ requires that $E_0\, k = B_0\, \omega/c$.

The exact same calculation shows the final Maxwell equation $\vec{\nabla} \times \vec{\mathbf{B}} = (1/c)\, \partial\vec{\mathbf{E}}/\partial t$ gives:

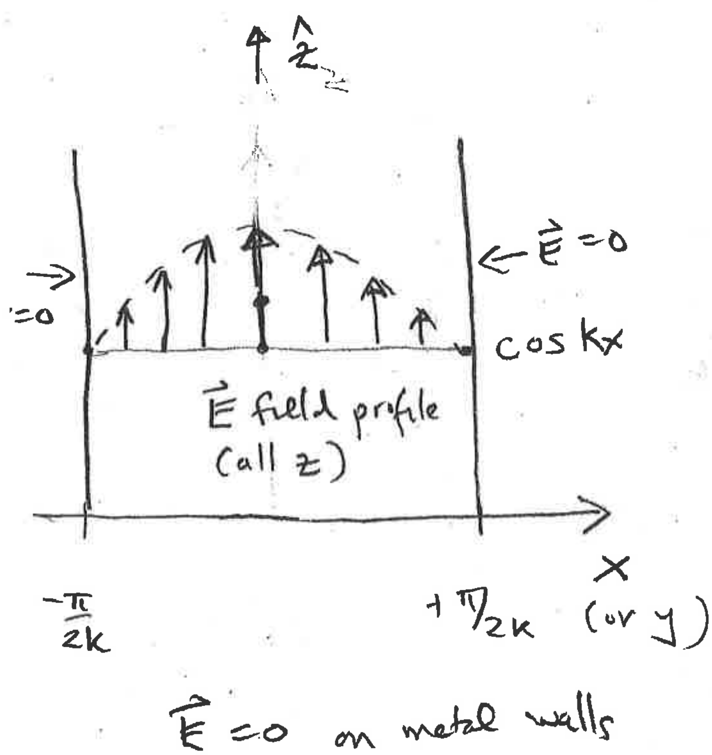
$$\vec{\nabla} \times \vec{\mathbf{B}} = -\hat{\mathbf{x}}\frac{\partial B_y}{\partial z} - +\hat{\mathbf{y}}\frac{\partial B_x}{\partial z} + \hat{\mathbf{z}}\Big(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\Big)$$
$$= B_0\, \hat{\mathbf{z}} \sin\omega t \Big(- k \cos kx \cos ky - k \cos kx \cos ky\Big)$$
$$\frac{\partial \vec{\mathbf{E}}}{\partial t} = -E_0\, \omega \sin\omega t\, \hat{\mathbf{z}}\Big(\cos kx \cos ky\Big) \tag{4}$$

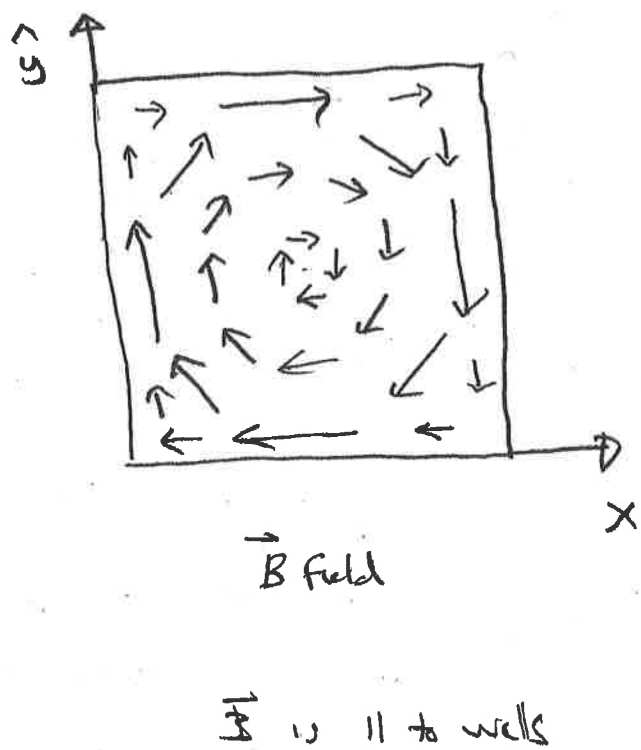
This yields a somewhat surprising result: $2B_0\,k = E_0\,\omega/c$. Putting this together with $E_0\,k = B_0\,\omega/c$ tells us that $\omega = \sqrt{2}\,kc$ and $E_0 = \sqrt{2}\,B_0$.

This weird appearance of $\sqrt{2}$ can also be derived by looking at the wave equation for E_z:

$$\frac{\partial^2 E_z}{\partial t^2} = -E_0\,\omega^2 \cos kx \cos ky \cos\omega t$$
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_z = -2E_0\,k^2 \cos kx \cos ky \cos\omega t \tag{5}$$

Again implying $\omega/c = \sqrt{2}\,k$.





2 A typical density of electrons in a metal is

$$n \approx 8.48 \cdot 10^{28} \ e^-/m^3 \quad \text{(copper)}$$

From this we get $k_F = (3\pi^2 n)^{1/3} \approx 1.36 \cdot 10^{10} \ m^{-1}$

and $E_f = \hbar^2 k_F^2/2m \approx \frac{(1.05 \cdot 10^{-34})^2 (1.36 \cdot 10^{10})^2}{9.11 \cdot 10^{-31}} \approx 2.23 \cdot 10^{-18}$ J

about 10 eV

The interaction energy ,

$$e^2/4\pi\epsilon_0 b \approx (1.6 \cdot 10^{-19})^2 / 4\pi(8.85 \cdot 10^{-12})(2.31 \cdot 10^{-9}) \approx 10^{-19}$$

where b is a typical separation set by

$$nb^3 \approx 1 \qquad b \approx n^{-1/3} \approx 2.31 \cdot 10^{-9}$$

So $E_f \gg$ Coulomb Energy

More crucially, note $E_f \sim n^{2/3}$ ← So grows more rapidly with n

Coulomb $\sim n^{1/3}$

KE wins at large density!

(a surprise!)