PHYSICS 200C, SPRING 2017
ELECTRICITY AND MAGNETISM

Assignment Two, Due Friday, April 21, 5:00 pm.

Problem numeration is from "Classical Electrodynamics", J.D. Jackson, third edition.

[1.] Jackson 6.3.

[2.] Jackson 6.4.

[3.] Jackson 6.8.

[4.] Show that if magnetic monopoles *did* exist, but all particles in nature had the same ratio of magnetic to electric charge, then one could redefine the sources and fields of electromagnetism so that the usual Maxwell equations (in which there are no magnetic monopoles) would emerge.

Physics 200 C
Spring 2017
Assignment 2 Sol'ns

1 Jackson 6.3

6.3 The homogeneous diffusion equation (5.160) for the vector potential for quasi-static fields in unbounded conducting media has a solution to the initial value problem of the form

$$\vec{A}(\vec{x},t) = \int d^3x' G(\vec{x}-\vec{x}',t)\vec{A}(\vec{x}',0)$$

where $\vec{A}(\vec{x}',0)$ describes the initial field configuration and G is an appropriate kernel.

a) Solve the initial value problem by use of a three-dimensional Fourier transform in space for $\vec{A}(\vec{x},t)$. With the usual assumption on intrchange of orders of integration, show that the Green function has the Fourier representation

$$G(\vec{x}-\vec{x}',t) = \frac{1}{(2\pi)^3}\int d^3k\, e^{-k^2t/\mu\sigma} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

and it is assumed that $t > 0$.

We define the Fourier transform by

$$\vec{A}(\vec{x},t) = \frac{1}{(2\pi)^3}\int \vec{A}(\vec{k},t)e^{-\vec{k}\cdot\vec{x}}\, d^3k$$

In this case, the diffusion equation $\nabla^2\vec{A} = \mu\sigma\partial\vec{A}/\partial t$ becomes

$$|i\vec{k}|^2\vec{A} = \mu\sigma\frac{\partial}{\partial t}\vec{A} \qquad \Rightarrow \qquad \frac{\partial}{\partial t}\vec{A} = -\frac{k^2}{\mu\sigma}\vec{A}$$

This equation is first order in time, and is easily solved to yield

$$\vec{A}(\vec{k},t) = \vec{A}(\vec{k},0)e^{-k^2t/\mu\sigma} \tag{1}$$

Note that we have written the solution in terms of initial conditions specified as $\vec{A}(\vec{k},0)$ at time $t=0$. This is essentially the answer in momentum space. All we have to do is to invert the transform. The inverse transform gives

$$\vec{A}(\vec{x},t) = \frac{1}{(2\pi)^3}\int \vec{A}(\vec{k},0)e^{-k^2t/\mu\sigma}e^{i\vec{k}\cdot\vec{x}}\, d^3k$$

where

$$\vec{A}(\vec{k},0) = \int \vec{A}(\vec{x}',0)e^{-i\vec{k}\cdot\vec{x}'}\, d^3x'$$

The result is

$$\vec{A}(\vec{x},t) = \frac{1}{(2\pi)^3}\int\int \vec{A}(\vec{x}',0)e^{-k^2t/\mu\sigma}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,d^3k\,d^3x'$$
$$= \int G(\vec{x}-\vec{x}',t)\vec{A}(\vec{x}',0)\,d^3x'$$

with the Greens function

$$G(\vec{x}-\vec{x}',t) = \frac{1}{(2\pi)^3}\int e^{-k^2t/\mu\sigma}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,d^3k$$

Alternatively, we could have noted directly from (1) that the solution in momentum space is a product of $e^{-k^2t/\mu\sigma}$ with $\vec{A}(\vec{k},0)$. As a result, the ordinary space solution is a convolution as indicated.

b) By introducing a Fourier decomposition in both space and time, and performing the frequency integral in the complex ω plane to recover the result of part *a*), show that $G(\vec{x}-\vec{x}',t)$ is the diffusion Green function that satisfies the inhomogeneous equation

$$\frac{\partial G}{\partial t} - \frac{1}{\mu\sigma}\nabla^2 G = \delta^{(3)}(\vec{x}-\vec{x}')\delta(t)$$

and vanishes for $t<0$.

Introducing the Fourier transform

$$G(\vec{x},t) = \frac{1}{(2\pi)^4}\int G(\vec{k},\omega)e^{i(\vec{k}\cdot\vec{x}-\omega t)}\,d^3k\,d\omega$$

the above inhomogeneous equation becomes

$$[(-i\omega)^2 - |i\vec{k}|^2/\mu\sigma]G = e^{-i\vec{k}\cdot\vec{x}'}$$

which gives the Greens' function

$$G(\vec{k},\omega) = \frac{e^{-i\vec{k}\cdot\vec{x}'}}{k^2/\mu\sigma - i\omega}$$

We may invert the transform by first performing the ω integral. We have

$$G(\vec{k},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\vec{k},\omega)e^{-i\omega t}\,d\omega = \frac{ie^{-i\vec{k}\cdot\vec{x}'}}{2\pi}\int\frac{e^{-i\omega t}}{\omega + ik^2/\mu\sigma}\,d\omega$$

This may be performed by contour integration. For $t>0$, Jordan's lemma tells us to close the contour in the lower-half plane. As a result, we pick up the residue at $\omega = -ik^2/\mu\sigma$. The result is

$$G(\vec{k},t) = (-2\pi i)\frac{ie^{-i\vec{k}\cdot\vec{x}'}}{2\pi}e^{-k^2t/\mu\sigma} = e^{-k^2t/\mu\sigma}e^{-i\vec{k}\cdot\vec{x}'}$$

On the other hand, for $t < 0$, we close the contour in the upper-half plane and end up with $G = 0$ as there are no enclosed poles. Finally, writing out the momentum space Fourier transform gives

$$G(\vec{x} - \vec{x}\,', t) = \frac{\Theta(t)}{(2\pi)^3} \int e^{-k^2 t/\mu\sigma} e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}\, d^3k \tag{2}$$

c) Show that if σ is uniform throughout all space, the Green function is

$$G(\vec{x}, t; \vec{x}\,', 0) = \Theta(t) \left(\frac{\mu\sigma}{4\pi t}\right)^{3/2} \exp\left(\frac{-\mu\sigma|\vec{x} - \vec{x}\,'|^2}{4t}\right)$$

Actually, we must take both μ and σ to be uniform in all space. In this case, the momentum integration in (2) may be performed by completing the square

$$\begin{aligned} G(\vec{x} - \vec{x}\,', t) &= \frac{\Theta(t)}{(2\pi)^3} e^{-\mu\sigma|\vec{x}-\vec{x}\,'|^2/4t} \int e^{-t|\vec{k} - i\mu\sigma(\vec{x}-\vec{x}\,')/2t|^2/\mu\sigma}\, d^3k \\ &= \frac{\Theta(t)}{(2\pi)^3} \left(\frac{\pi\mu\sigma}{t}\right)^{3/2} e^{-\mu\sigma|\vec{x}-\vec{x}\,'|^2/4t} \\ &= \Theta(t) \left(\frac{\mu\sigma}{4\pi t}\right)^{3/2} e^{-\mu\sigma|\vec{x}-\vec{x}\,'|^2/4t} \end{aligned} \tag{3}$$

d) Suppose that at time $t' = 0$, the initial vector potential $\vec{A}(\vec{x}\,', 0)$ is nonvanishing only in a localized region of linear extent d around the origin. The time dependence of the fields is observed at a point P far from the origin, i.e., $|\vec{x}\,| = r \gg d$. Show that there are three regimes of time, $0 < t \leq T_1$, $T_1 \leq t \leq T_2$, and $t \gg T_2$. Give plausible definitions of T_1 and T_2, and describe qualitatively the time dependence at P. Show that in the last regime, the vector potential is proportional to the volume integral of $\vec{A}(\vec{x}\,', 0)$ times $t^{-3/2}$, assuming that integral exists. Relate your discussion to those of Section 5.18.B and Problems 5.35 and 5.36.

For a local 'disturbance' near the origin, physical intuition tells us that it will take some time before the observer at point P will manage to observe it. For a diffusion problem, this time is essentially the timescale for diffusion, set by the diffusion coefficient $D = 1/\mu\sigma$, where the diffusion equation is of the form $\partial\rho/\partial t = \vec{\nabla}\cdot(D\vec{\nabla}\rho)$. To be specific, the field at point P is given by the convolution

$$\begin{aligned} \vec{A}(\vec{x}, t) &= \int G(\vec{x} - \vec{x}\,', t)\vec{A}(\vec{x}\,', 0)\, d^3x' \\ &= \Theta(t) \left(\frac{\mu\sigma}{4\pi t}\right)^{3/2} \int \vec{A}(\vec{x}\,', 0) e^{-\mu\sigma|\vec{x}-\vec{x}\,'|^2/4t}\, d^3x' \end{aligned}$$

Assuming $|\vec{x}\,| = r \gg d$, we may approximate the integral by simply taking $|\vec{x} - \vec{x}\,'|^2 \approx d^2$. This gives

$$\vec{A}(r, t) \approx \Theta(t) \left(\frac{\mu\sigma d}{4\pi t}\right)^{3/2} e^{-\mu\sigma r^2/4t} \langle\vec{A}\rangle_{t=0} \tag{4}$$

where $\langle \vec{A} \rangle$ is the spatial average of $\vec{A}$ in the nonvanishing region. Defining $\tau_1 = \mu\sigma r^2/4$ and $\tau_2 = \mu\sigma d/4\pi$, the behavior of the vector potential is then

$$\vec{A}(r,t) \approx \Theta(t) \left(\frac{\tau_2}{t}\right)^{3/2} e^{-\tau_1/t}$$

At short times, $t \ll \tau_1$, the exponential is highly suppressed, and there is no signal at point P. After time τ_1, however, the exponential becomes of order $\mathcal{O}(1)$. The initial vector potential has now diffused to P. However, because of the $(\tau_2/t)^{3/2}$ prefactor, the signal dies away at long times. For a rough estimate, we take

$$T_1 = \tau_1 = \mu\sigma r^2/4 \qquad T_2 = 2T_1$$

Then for $t < T_1$ the vector potential at point P is essentially zero. Between T_1 and T_2, the vector potential is non-zero, and at long times after T_2, everything has diffused away. Finally, noting that the volume integral of $\vec{A}(\vec{x}\,', 0)$ is simply $d^3\langle \vec{A} \rangle$, the expression in (4) demonstrates that at late times (when the exponential is essentially unity) the vector potential is indeed proportional to this volume integral times $t^{-3/2}$.

2 Jackson 6.4

6.4 A uniformly magnetized and conducting sphere of radius R and total magnetic moment $m = 4\pi M R^3/3$ rotates about its magnetization axis with angular speed ω. In the steady state no current flows in the conductor. The motion is nonrelativistic; the sphere has no excess charge on it.

a) By considering Ohm's law in the moving conductor, show that the motion induces an electric field and a uniform volume charge density in the conductor, $\rho = -m\omega/\pi c^2 R^3$.

We assume the sphere is magnetized and spinning along the $\hat{z}$ axis. Since the magnetic moment is $\vec{m} = \vec{M}V$ where $V = \frac{4}{3}\pi R^3$ is the volume of the sphere, we see that the magnetization is simply $\vec{M} = M\hat{z}$. As demonstrated earlier, a uniformly magnetized sphere has a uniform magnetic induction $\vec{B} = \frac{2}{3}\mu_0 \vec{M}$ in its interior. In terms of m, this is

$$\vec{B} = \tfrac{2}{3}\mu_0 \left(\frac{3}{4\pi R^3} m\hat{z}\right) = \frac{\mu_0 m}{2\pi R^3}\hat{z}$$

We now observe that the electric field $\vec{E}'$ in the rotating frame of the sphere may be related to lab quantities by $\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$. Ohm's law in the rotating reference frame is then $\vec{J} = \sigma\vec{E}' = \sigma(\vec{E} + \vec{v} \times \vec{B})$. Since no current flows in the steady state ($\vec{J} = 0$), this motion must induce an electric field $\vec{E} = -\vec{v} \times \vec{B}$. Using $\vec{\omega} = \omega\hat{z}$ and $\vec{v} = \omega \times \vec{r}$, we obtain

$$\vec{E} = -(\vec{\omega} \times \vec{r}) \times \vec{B} = -\frac{\mu_0 m\omega}{2\pi R^3}(\hat{z} \times \vec{r}) \times \hat{z} = -\frac{\mu_0 m\omega}{2\pi R^3}(\vec{r} - \hat{z}(\hat{z} \cdot \vec{r}))$$

The vector structure is essentially a projection of $\vec{r}$ into the horizontal plane perpendicular to the $\hat{z}$ axis. In cylindrical coordinates, this indicates that

$$E_\rho = -\frac{\mu_0 m\omega\rho}{2\pi R^3} \tag{5}$$

It is then a simple matter of applying Gauss' law to recover the volume charge density. However, before we do so, we note that this is a cylindrically symmetric electric field (pointed horizontally inward towards the $\hat{z}$ axis). It may at first be somewhat surprising that a sphere will give a cylindrical electric field. However, rotation about an axis is actually a cylindrical process. So from this point of view, the electric field is quite natural.

Using $\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}$ we obtain a uniform volume charge density

$$\rho = \epsilon_0 \frac{\partial E_\rho}{\partial \rho} = -\frac{\mu_0 \epsilon_0 m\omega}{2\pi R^3} = -\frac{m\omega}{2\pi c^2 R^3}$$

It is important to note that, while the charge density is uniform inside the sphere, the electric field is *not* radial. The discrepancy between a uniform spherical charge distribution and the cylindrical electric field must then arise due to a surface charge. This then provides a hint as to how to approach the remainder of this problem.

b) Because the sphere is electrically neutral, there is no monopole electric field outside. Use symmetry arguments to show that the lowest possible electric multipolarity is quadrupole. Show that only a quadrupole field exists outside and that the quadrupole moment tensor has nonvanishing components, $Q_{33} = -4m\omega R^2/3c^2$, $Q_{11} = Q_{22} = -Q_{33}/2$.

No charge resides outside the sphere. As a result, the exterior field may be described through the multipole expansion. As indicated, charge neutrality guarantees the vanishing of the monopole ($l = 0$) moment. Furthermore, the odd moments vanish due to symmetry of the electric field (5) under the parity transformation $z \to -z$. (That is of course the internal field; however we may see that the external field must necessarily respect the symmetry of the internal one.) Thus a simple symmetry argument demonstrates that the lowest possible multipole is the quadrupole ($l = 2$). Symmetry along will not preclude higher even moments. However an explicit calculation will.

Without knowing the surface charge, we cannot directly calculate the electric multipoles. However, we note that the interior electric field (5) can be integrated to obtain the interior electrostatic potential

$$\Phi(\rho) = -\int \vec{E} \cdot d\vec{\ell} = -\int E_\rho \, d\rho = \Phi_0 + \frac{\mu_0 m\omega\rho^2}{4\pi R^3}$$

Converted back to spherical coordinates, this gives

$$\Phi(r, \theta) = \Phi_0 + \frac{\mu_0 m\omega}{4\pi R^3} r^2 \sin^2\theta = \Phi_0 + \frac{\mu_0 m\omega}{6\pi R^3} r^2 [P_0(\cos\theta) - P_2(\cos\theta)]$$

where we have converted $\sin^2\theta$ into Legendre polynomials. This can be written explicitly as a Legendre expansion

$$\Phi(r,\theta) = \left(\Phi_0 + \frac{\mu_0 m\omega}{6\pi R^3}r^2\right)P_0(\cos\theta) - \frac{\mu_0 m\omega}{6\pi R^3}r^2 P_2(\cos\theta)$$

so that in particular the potential at the surface of the sphere is

$$\Phi(R,\theta) = \left(\Phi_0 + \frac{\mu_0 m\omega}{6\pi R}\right)P_0(\cos\theta) - \frac{\mu_0 m\omega}{6\pi R}P_2(\cos\theta)$$

We may now solve for the exterior potential by treating this as an electrostatic boundary value problem. We recall that, given a sphere with azimuthally symmetric potential $V(\theta) = \sum_l \alpha_l P_l(\cos\theta)$ on the surface, the exterior solution has the form $\Phi(r,\theta) = \sum_l \alpha_l (R/r)^{l+1} P_l(\cos\theta)$. Furthermore, charge neutrality in the present case forces the monopole ($l = 0$) term to vanish. Hence we find that $\Phi_0 = -\mu_0 m\omega/6\pi R$, and that the external potential is

$$\Phi(r,\theta) = -\frac{\mu_0 m\omega R^2}{6\pi r^3}P_2(\cos\theta) \tag{6}$$

Incidentally, we could write an expression valid both in the interior and the exterior as

$$\Phi(r,\theta) = \frac{\mu_0 m\omega}{6\pi R}\left[\left(\frac{r^2}{R^2} - 1\right)\Theta(R-r)P_0(\cos\theta) - R\frac{r_<^2}{r_>^3}P_2(\cos\theta)\right] \tag{7}$$

Note that this potential is only harmonic outside the sphere; inside the sphere the r^2/R^2 term multiplying $P_0(\cos\theta)$ is not of the right $(A_l r^l + B_l r^{-l-1})P_l(\cos\theta)$ form to be harmonic. However, this is present precisely because of the uniform volume charge density, which acts as a $l = 0$ source.

In any case, we are essentially done, as the exterior potential (6) clearly has only a quadrupole term

$$\Phi = -\sqrt{\frac{4\pi}{5}}\frac{\mu_0 m\omega R^2}{6\pi}\frac{Y_{2,0}(\theta,\phi)}{r^3}$$

Comparing this with the multipole expansion

$$\Phi = \frac{1}{4\pi\epsilon_0}\sum_{l,m}\frac{4\pi}{2l+1}q_{l,m}\frac{Y_{l,m}(\theta,\phi)}{r^{l+1}}$$

gives

$$q_{2,0} = -4\pi\epsilon_0\sqrt{\frac{5}{4\pi}}\frac{\mu_0 m\omega R^2}{6\pi} = -\sqrt{\frac{5}{4\pi}}\frac{2m\omega R^2}{3c^2}$$

Converting to cartesian tensors yields

$$Q_{33} = 2\sqrt{\frac{4\pi}{5}}q_{2,0} = -\frac{4m\omega R^2}{3c^2}, \qquad Q_{11} = Q_{22} = -\tfrac{1}{2}Q_{33}$$

c) By considering the radial electric fields inside and outside the sphere, show that the necessary surface-chrage density $\sigma(\theta)$ is

$$\sigma(\theta) = \frac{1}{4\pi R^2}\frac{4m\omega}{3c^2}\left[1 - \frac{5}{2}P_2(\cos\theta)\right]$$

The surface charge may be computed by first obtaining the jump in the normal component of the electric field at the surface of the sphere. Working in spherical components, and taking the gradient of the potential (7), we find

$$E_r^{\rm out} = \frac{\mu_0 m\omega R^2}{2\pi r^4}P_2(\cos\theta)$$
$$E_r^{\rm in} = -\frac{\mu_0 m\omega r}{3\pi R^3}[P_0(\cos\theta) - P_2(\cos\theta)]$$

The surface charge is then computed as

$$\sigma = \epsilon_0(E_r^{\rm out} - E_r^{\rm in})\Big|_{r=R} = \frac{\mu_0\epsilon_0 m\omega}{3\pi R^2}[\tfrac{3}{2}P_2(\cos\theta) - (P_0(\cos\theta) - P_2(\cos\theta))]$$
$$= \frac{m\omega}{3\pi c^2 R^2}[P_0(\cos\theta) - \tfrac{5}{2}P_2(\cos\theta)]$$

d) The rotating sphere serves as a unipolar induction device if a stationary circuit is attached by a slip ring to the pole and a sliding contact to the equator. Show that the line integral of the electric field from the equator contact to the pole contact by any path) is $\mathcal{E} = \mu_0 m\omega/4\pi R$.

Although the sphere is rotating, both the electric and the magnetic field are static. Hence the line integral of the electric field gives simply the electrostatic potential. In this case

$$\mathcal{E} = \int_{\rm equator}^{\rm pole}\vec{E}\cdot d\vec{\ell} = \Phi_{\rm equator} - \Phi_{\rm pole} = \Phi(\theta = \tfrac{\pi}{2}) - \Phi(\theta = 0)$$

Using (6) or (7) evaluated on the surface, this becomes

$$\mathcal{E} = -\frac{\mu_0 m\omega}{6\pi R}[P_2(0) - P_2(1)] = \frac{\mu_0 m\omega}{4\pi R}$$

$$\frac{}{3}\mu_0 M(R\cos\theta\vec{\omega} - \omega\vec{r})\cdot(Rd\theta\hat{\theta})$$

$$= \frac{2}{3}\mu_0 M\omega R^2 \int_0^{\pi/2} \sin\theta\cos\theta d\theta = \frac{1}{3}\mu_0 M\omega R^2 = \frac{\mu_0 m\omega}{4\pi R}$$

Additional stuff for my record

The potential due to the surface charge

$$\Phi_\sigma(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint \frac{\sigma(\theta')}{|\vec{r}-\vec{r'}|}da' = \frac{R^2}{2\epsilon_0}\int \sigma(\theta')\sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\theta')P_\ell(\cos\theta)d\cos\theta'$$

$$= \frac{R^2\sigma_0}{2\epsilon_0}\sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\theta)\int P_\ell(\cos\theta')\left\{1-\frac{5}{2}P_2(\cos\theta')\right\}d\cos\theta'$$

$$= \frac{R^2\sigma_0}{2\epsilon_0}\left\{\frac{2}{r_>} - \frac{r_<^2}{r_>^3}P_2(\cos\theta)\right\} = \frac{\mu_0}{3\pi}m\omega\left\{\frac{1}{r_>} - \frac{r_<^2}{r_>^3}\frac{P_2(\cos\theta)}{2}\right\}$$

The total potential outside the sphere ($r_< = a$ and $r_> = r$):

$$\Phi(\vec{r}) = \Phi_\rho(\vec{r}) + \Phi_\sigma(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q_\rho}{r} + \frac{R^2\sigma_0}{\epsilon_0 r} - \frac{a^4\sigma_0}{2\epsilon_0 r^3}P_2(\cos\theta) = -\frac{m\omega R^2}{6\pi c^2\epsilon_0}\frac{P_2(\cos\theta)}{r^3} = -\frac{\mu_0}{6\pi}\frac{R^2}{r^3}P_2(\cos\theta)$$

The electric field outside the sphere:

$$\vec{E}(\vec{r}) = -\nabla\Phi(\vec{r}) = -\frac{\mu_0}{4\pi}\frac{m\omega R^2}{}$$

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3 Jackson 6.8

$$\frac{}{R^2}(1 - P_2(\cos\theta))\}$$

Problem 6.8

In an external uniform electric field $\vec{E}$, the sphere is uniformly polarized with the polarization given by Eq. (4.57):

$$\vec{P} = 3\epsilon_0\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\vec{E_0}$$

Therefore, the bound volume and surface charge densities are:

$$\rho_b = -\nabla\cdot\vec{P} = 0, \qquad \sigma_b = \vec{P}\cdot\vec{n}$$

where $\vec{n}$ is the normal vector on the sphere surface. Since the sphere is rotating, the bound surface charge results an effective surface current with density:

$$\vec{K}_M = \sigma_b\vec{v} = (\vec{P}\cdot\vec{n})(\vec{\omega}\times\vec{r})|_{r=a} = a(\vec{P}\cdot\vec{n})(\omega\times\vec{n})$$

Comparing with the effective surface current density $\vec{K}_M = \vec{M} \times \vec{n}$ due to magnetization $\vec{M}$, we identify $a(\vec{P}\cdot\vec{n})\vec{\omega}$ as an effective magnetization. Therefore, the effective magnetic surface charge density

$$\sigma_M(\theta,\phi) = \vec{M}_{\text{eff.}} \cdot \vec{n} = a(\vec{P}\cdot\vec{n})(\vec{\omega}\cdot\vec{n}) = 3\epsilon_0 a \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}(\vec{E}_0\cdot\vec{n})(\omega\cos\theta) = 3\epsilon_0 \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0} a\omega E_0 \sin\theta\cos\theta\cos\phi$$

The magnetic scalar potential $\Phi_M(\vec{r})$ (Eq. (5.100)):

$$\Phi_M(\vec{r}) = \frac{1}{4\pi}\oint \frac{\sigma_M}{|\vec{r}-\vec{r}'|}da' = \frac{3}{4\pi}\epsilon_0 \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0} a^3 \omega E_0 \int \frac{\sin\theta'\cos\theta'\cos\phi'}{|\vec{r}-\vec{r}'|}d\Omega'$$

Using the identity:

$$\sin\theta'\cos\theta'\cos\phi' = -\sqrt{\frac{8\pi}{15}} Re\left\{Y_{21}(\theta',\phi')\right\}$$

and expanding $1/|\vec{r}-\vec{r}'|$ using spherical harmonics, the integral becomes:

$$\int \frac{\sin\theta'\cos\theta'\cos\phi'}{|\vec{r}-\vec{r}'|}d\Omega' = -\sqrt{\frac{8\pi}{15}} Re\left\{\sum_{\ell,m} \frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta,\phi)\int Y^*_{\ell m}(\theta',\phi')Y_{21}(\theta',\phi')d\Omega'\right\}$$

$$= -\sqrt{\frac{8\pi}{15}} Re\left\{\sum_{\ell,m} \frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta,\phi)\delta_{\ell,2}\delta_{m,1}\right\}$$

$$= \frac{4\pi}{5}\frac{r_<^2}{r_>^3}\left\{-\sqrt{\frac{8\pi}{15}} Re\{Y_{21}(\theta,\phi)\}\right\}$$

$$= \frac{4\pi}{5}\frac{r_<^2}{r_>^3}\sin\theta\cos\theta\cos\phi$$

where $r_< = \min(r,a)$ and $r_> = \max(r,a)$. Therefore, the scalar potential

$$\Phi_M(\vec{r}) = \frac{1}{4\pi}\oint \frac{\sigma_M}{|\vec{r}-\vec{r}'|}da' = \frac{3}{4\pi}\epsilon_0 \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0} a^3 \omega E_0 \left\{\frac{4\pi}{5}\frac{r_<^2}{r_>^3}\sin\theta\cos\theta\cos\phi\right\}$$

$$= \frac{3}{5}\epsilon_0 \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\omega E_0 \frac{a^3 r_<^2}{r^2 r_>^3}(r\sin\theta\cos\phi)(r\cos\theta)$$

Note

$$\frac{a^3 r_<^2}{r^2 r_>^3} = \frac{a^3 r_<^2 r_>^2}{r^2 r_>^5} = \frac{a^3 r^2 a^2}{r^2 r_>^5} = \left\{\frac{a}{r_>}\right\}^5$$

The magnetic field $\vec{H}$ can be determined from $\Phi_M(\vec{r})$:

$$\Phi_M(\vec{r}) = \frac{3}{5}\epsilon_0\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\omega E_0 \left\{\frac{a}{r_>}\right\}^5 \cdot xz = \frac{1}{5}P\omega\left\{\frac{a}{r_>}\right\}^5 \cdot xz$$

What if the electric field is along the rotational axis?

The effective magnetic surface charge density:

$$\sigma_M(\theta,\phi) = a(\vec{P}\cdot\vec{n})(\vec{\omega}\cdot\vec{n}) = 3\epsilon_0\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}a\omega E_0\cos^2\theta$$

$$\Phi_M(\vec{r}) = \frac{1}{4\pi}\oint \frac{\sigma_M}{|\vec{r}-\vec{r}'|}da' = \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}a^3\omega\epsilon_0 E_0\left\{\frac{1}{r_>}+\frac{2}{5}\frac{r_<^2}{r_>^3}P_2(\cos\theta)\right\}$$

(Not assigned)

Problem 6.11

(a) The momentum conservation equation

$$\frac{d}{dt}(\vec{P}_{\text{fields}}+\vec{P}_{\text{mech.}}) = \oint \sum_j T_{ij}n_j da = -\oint(-\vec{\vec{T}})\cdot\vec{n}da$$

implies that the projection of the momentum flow along the direction of $\vec{n}$ is given by $-\vec{\vec{T}}\cdot\vec{n}$ where $\vec{\vec{T}}$ is the Maxwell stress (momentum) tensor:

$$T_{ij} = \epsilon_0(E_iE_j - \frac{1}{2}E^2\delta_{ij}) + \mu_0(H_iH_j - \frac{1}{2}H^2\delta_{ij})$$

Physically $-T_{ij}$ is the rate at which the i^{th}-component of the momentum is crossing a unit area in the $j^{th}-$direction. In a Cartesian coordinate system with the $z-$axis along the wave propagation direction and $\vec{E}$ along the $x-$direction:

$$\vec{E} = E\hat{x}, \quad \vec{H} = H\hat{y}$$

The i^{th} component of the linear momentum flowing into the surface (in the direction $\vec{n}=\hat{z}$) per unit time per unit cross section is therefore

$$p_i = \sum_j(-T_{ij})n_j = -T_{i3} = -\epsilon_0(E_iE_3 - \frac{1}{2}E^2\delta_{i,3}) - \mu_0(H_iH_3 - \frac{1}{2}H^2\delta_{i,3}) = \frac{1}{2}(\epsilon_0E^2+\mu_0H^2)\delta_{i,3}$$

In the chosen coordinate system, the only non-vanishing component is p_z. The force exerted on the wave from the surface per unit area (according to Newton's second law):

$$F_z = \Delta p_z = (0-p_z) = -\frac{1}{2}(\epsilon_0E^2+\mu_0H^2)$$

Therefore, the radiation pressure on the surface (Newton's third law):

$$P_z = -F_z = \frac{1}{2}(\epsilon_0E^2+\mu_0H^2)$$

which is the energy density in the electromagnetic wave.

(Not assigned)

Problem 6.13

(a) Note: only need to work out the first non-zero terms in electric/magnetic fields. To a good approximation, the conductors are at equipotential and have uniform surface charge distributions. Choose a Cartesian coordinate system with its origin at the center of the capacitor, the $x-$axis parallel to the edge a and pointing to the current feed, the $y-$ axis perpendicular to the two planes. Let $Q(t) = Q_0e^{-i\omega t}$ be the total charge on the bottom plate, the electric field in between the plates is therefore

$$\vec{E}(\vec{r},t) = \frac{\sigma(t)}{\epsilon_0}\hat{y} = \frac{1}{\epsilon_0}\frac{Q_0}{ab}e^{-i\omega t}\hat{y}$$

The charge on the $x' < x$ portion of the bottom plate is:

$$Q(\vec{r},t) = b(x+\frac{a}{2})\sigma(t) = \frac{Q_0}{ab}e^{-i\omega t}b(x+\frac{a}{2}) = (\frac{1}{2}+\frac{x}{a})Q_0e^{-i\omega t}$$

The surface current density

$$\vec{K}(\vec{r},t) = -\frac{1}{b}\frac{\partial Q(\vec{r},t)}{\partial t}\hat{x} = i\omega(\frac{1}{2}+\frac{x}{a})\frac{Q_0}{b}e^{-i\omega t}\hat{x}$$

4. We wish to show that if magnetic monopoles did exist, but all particles in Nature had the same ratio of magnetic to electric charge, then one could redefine the sources and fields of electromagnetism so that the usual Maxwell equations would emerge.

The usual Maxwell equations are

$$\vec{\nabla}\cdot\vec{D} = \rho_e, \qquad \vec{\nabla}\times\vec{H} = \vec{J}_e + \frac{\partial \vec{D}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0, \qquad \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

If there exist magnetic charge and current densities, ρ_m and $\vec{J}_m$, in addition to the electric charge and current densities, ρ_e and $\vec{J}_e$, the Maxwell equations would become

$$\vec{\nabla}\cdot\vec{D} = \rho_e, \qquad \vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_e \qquad (*)$$
$$\vec{\nabla}\cdot\vec{B} = \rho_m, \qquad -\vec{\nabla}\times\vec{E} = \frac{\partial \vec{B}}{\partial t} + \vec{J}_m$$

We consider $\frac{\rho_m}{\rho_e}$ to be fixed: $\frac{\rho_m}{\rho_e} = c$ (Note that this implies that we are in Gaussian units: $[\rho_m]=[\rho_e]$. In SI units, ρ_m differs in dimensions from ρ_e) (In Gaussian units, the impedance of free space $Z_0 = \sqrt{\mu_0/\epsilon_0} \longrightarrow 1$)

Let us consider the following duality transformation of the EM fields:

$$\vec{E} = \vec{E}'\cos\xi + \vec{H}'\sin\xi, \quad \vec{D} = \vec{D}'\cos\xi + \vec{B}'\sin\xi \qquad (1)$$
$$\vec{H} = -\vec{E}'\sin\xi + \vec{H}'\cos\xi, \quad \vec{B} = -\vec{D}'\sin\xi + \vec{B}'\cos\xi,$$

where ξ is a real (pseudoscalar) angle.

Suppose that we transform the sources in the same way

$$\rho_e = \rho_e'\cos\xi + \rho_m'\sin\xi, \qquad \vec{J}_e = \vec{J}_e'\cos\xi + \vec{J}_m'\sin\xi \qquad (2)$$
$$\rho_m = -\rho_e'\sin\xi + \rho_m'\cos\xi, \qquad \vec{J}_m = -\vec{J}_e'\sin\xi + \vec{J}_m'\cos\xi$$

This duality transformation leaves the Maxwell equations with magnetic monopoles invariant. The form for the primed fields, $\vec{E}', \vec{B}', \vec{D}', \vec{H}'$ is the same as for the original $\vec{E}, \vec{B}, \vec{D}, \vec{H}$.

Now, consider our assumption that <u>all</u> particles have the same ratio of ρ_m to ρ_e: $\frac{\rho_m}{\rho_e} = \text{const}$

We can then perform a duality transformation such that the monopole densities vanish. Namely, we choose a ξ such that

$$\begin{cases} \rho_m = 0, \\ \vec{J}_m = 0 \end{cases}$$

If we demand these conditions, we get

$$\rho_m = 0 = -\rho_e' \sin\xi^* + \rho_m' \cos\xi^* \Rightarrow \cot\xi^* = \frac{\rho_e'}{\rho_m'}$$

$$\text{or } \tan\xi^* = \frac{\rho_m'}{\rho_e'}$$

$$\Rightarrow \vec{J}_m = 0 \;:\; \vec{J}_m' = \vec{J}_e' \tan\xi^* \qquad \Rightarrow \boxed{\xi^* = \tan^{-1}\left(\frac{\rho_m'}{\rho_e'}\right)}$$

With this, we get the following set of equations for the unprimed fields: $\boxed{\begin{matrix} \vec{\nabla}\cdot\vec{D} = \rho_e, & \vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{J}_e \\ \vec{\nabla}\cdot\vec{B} = 0, & -\vec{\nabla}\times\vec{E} = \frac{\partial\vec{B}}{\partial t} \end{matrix}}$ These are the usual Maxwell equations.

Let us denote the angle that makes $\rho_m = 0$ as ξ^*. Then, for the primed fields we have:

$$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot\left(-\vec{B}'\sin\xi^* + \vec{B}'\cos\xi^*\right) = 0 \Rightarrow \vec{\nabla}\cdot\vec{B}' = \vec{\nabla}\cdot\vec{D}'\tan\xi^*$$

With this, we get: Since $\rho_m' = \rho_e' \tan\xi^*$,

$$\vec{\nabla}\cdot\vec{D} = \rho_e : \vec{\nabla}\cdot\left(\vec{D}'\cos\xi^* + \vec{B}'\sin\xi^*\right) = \rho_e'\cos\xi^* + \rho_m'\sin\xi^*$$

$$\vec{\nabla}\cdot\vec{D}'\left(\frac{1}{\cos\xi^*}\right) = \rho_e'\left(\cos\xi^* + \frac{\sin^2\xi^*}{\cos\xi^*}\right)$$

$$\Longrightarrow \boxed{\vec{\nabla}\cdot\vec{D}' = \rho_e'} \text{, as expected.}$$

Further, we have $\vec{\nabla}\cdot\vec{B}' = \vec{\nabla}\cdot\vec{D}'\tan\xi^* = \rho_e'\tan\xi^* = \rho_m'$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{B}' = \rho_m'}$$

$$\vec{\nabla}\times\vec{H} = \vec{\nabla}\times\left(-\vec{E}'\sin\xi^* + \vec{H}'\cos\xi^*\right) = \frac{\partial}{\partial t}\left(\vec{D}'\cos\xi^* + \vec{B}'\sin\xi^*\right) + \left(\vec{J}_e'\cos\xi^* + \vec{J}_m'\sin\xi^*\right)$$

Matching coefficients of $\sin\xi^*$, $\cos\xi^*$, we have that

$$\vec{\nabla}\times\vec{H}' = \frac{\partial \vec{D}'}{\partial t} + \vec{J}_e' \quad \text{and} \quad -\vec{\nabla}\times\vec{E}' = \frac{\partial \vec{B}'}{\partial t} + \vec{J}_m'$$

These look like the Maxwell equations with magnetic monopoles.

The point is that if we define the fields and sources as in ① and ②, respectively, under the duality transformation, then assuming that $\frac{\rho_m}{\rho_e}$ = const, we find that we can choose the angle ξ such that $\rho_m = 0$, $\vec{J}_m = 0$. With this choice, we recover the standard Maxwell equations. The primed 'dual' fields then obey the Maxwell equations with monopoles, while the unprimed fields obey the standard equations.

If we were to take the primed set and express the fields $\{\vec{E}', \vec{B}', \vec{D}', \vec{H}'\}$ in terms of $\{\vec{E}, \vec{B}, \vec{D}, \vec{H}\}$, and $\{\rho_e', \rho_m', \vec{J}_e', \vec{J}_m'\}$ in terms of $\{\rho_e, \vec{J}_e\}$, we would find that (going the other way)

$$\begin{cases} \vec{\nabla}\cdot\vec{D}' = \rho_e', & \vec{\nabla}\times\vec{H}' = \frac{\partial \vec{D}'}{\partial t} + \vec{J}_e' \\ \vec{\nabla}\cdot\vec{B}' = \rho_m', & -\vec{\nabla}\times\vec{E}' = \frac{\partial \vec{B}'}{\partial t} + \vec{J}_m' \end{cases}$$

becomes the standard for $\xi = \xi^*$

$$\vec{\nabla}\cdot\vec{D} = \rho_e, \quad \vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_e$$
$$\vec{\nabla}\cdot\vec{B} = 0, \quad -\vec{\nabla}\times\vec{E} = \frac{\partial \vec{B}}{\partial t}$$

Explicitly, if we write

$$\begin{cases} \vec{E}' = \cos\xi^*\,\vec{E} - \sin\xi^*\,\vec{H}, & \vec{D}' = \cos\xi^*\,\vec{D} - \sin\xi^*\,\vec{B} \\ \vec{B}' = \sin\xi^*\,\vec{D} + \cos\xi^*\,\vec{B}, & \vec{H}' = \sin\xi^*\,\vec{E} + \cos\xi^*\,\vec{H} \\ \rho_e' = \cos\xi^*\,\rho_e, \quad \vec{J}_e' = \cos\xi^*\,\vec{J}_e, & \rho_m' = \rho_e'\tan\xi^* = \sin\xi^*\,\rho_e \\ \vec{J}_m' = \vec{J}_e'\tan\xi^* = \sin\xi^*\,\vec{J}_e \end{cases}$$

$\Rightarrow \vec{\nabla}\cdot\vec{D}' = \rho_e' : \quad \vec{\nabla}\cdot(\cos\xi^*\,\vec{D} - \sin\xi^*\,\vec{B}) = \cos\xi^*\,\rho_e$.

Matching the $\cos\xi^*$ and $\sin\xi^*$ coefficients, get:

$$\begin{cases} \vec{\nabla}\cdot\vec{D} = \rho_e \\ \vec{\nabla}\cdot\vec{B} = 0 \end{cases}$$

$\Rightarrow \vec{\nabla}\times\vec{H}' = \vec{\nabla}\times(\sin\xi^*\,\vec{E} + \cos\xi^*\,\vec{H}) = \frac{\partial}{\partial t}(\cos\xi^*\,\vec{D} - \sin\xi^*\,\vec{B})$

Matching the $\cos\xi^*$ and $\sin\xi^*$ coefficients, get:

$$\begin{cases} \vec{\nabla}\times\vec{H} = \dfrac{\partial\vec{D}}{\partial t} + \vec{J}_e \\ -\vec{\nabla}\times\vec{E} = \dfrac{\partial\vec{B}}{\partial t} \end{cases}$$

These are indeed the standard Maxwell equations.