PHYSICS 200C, SPRING 2017
ELECTRICITY AND MAGNETISM
Midterm Exam

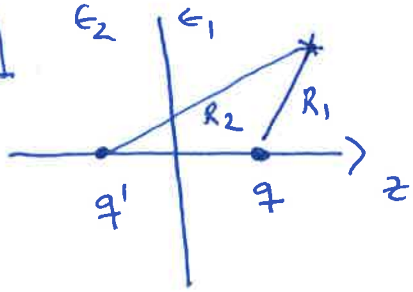
Instructions: Do *one* of problems 1 or 2; and *one* of problems 3 or 4.

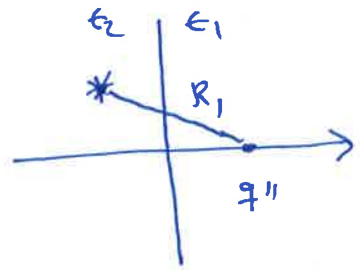
[1.] A point charge Q is situated at position $(x, y, z) = (0, 0, d)$. The dielectric constant is ϵ_1 for $z > 0$ and ϵ_2 for $z < 0$. Compute the electrostatic potential at all points in space.

-OR-

[2.] A dielectric sphere is placed in a uniform electric field. Compute the potential everywhere in space, and the electric field and polarization vectors inside the sphere.

$\boxed{1}$





For V at $z>0$ insert image charge q' at $(0,0,-d)$

For V at $z<0$ use charge q'' at $(0,0,+d)$

← You are allowed to put image charges any way you desire, as long as they are outside the region for which you are computing V!

$z>0$ $\quad V = \dfrac{q}{4\pi\epsilon_1 R_1} + \dfrac{q'}{4\pi\epsilon_1 R_2}$ $\qquad z<0$ $\quad V = q''/4\pi\epsilon_2 R_1$

tangential component of $\vec{E}$ is continuous at interface $z=0$

$$-\frac{1}{4\pi\epsilon_1}\frac{\partial}{\partial\rho}\left(\frac{q}{R_1}+\frac{q'}{R_2}\right)\Bigg|_{z=0} = -\frac{1}{4\pi\epsilon_2}\frac{\partial}{\partial\rho}\, q''/R_1\Big|_{z=0}$$

$\rho = \sqrt{x^2+y^2}$

$$1/\epsilon_1 \frac{q\rho + q'\rho}{(\rho^2+d^2)^{3/2}} = \frac{1}{\epsilon_2}\frac{q''\rho}{(\rho^2+d^2)^{3/2}} \Rightarrow \boxed{\frac{1}{\epsilon_1}(q+q') = \frac{1}{\epsilon_2}q''}$$

normal component of $\vec{D}$ is continuous at interface $z=0$

$$-\frac{1}{4\pi}\frac{\partial}{\partial z}\left(q/R_1 + q'/R_2\right)\Big|_{z=0} = -\frac{1}{4\pi}\frac{\partial}{\partial z}\, q''/R_1\Big|_{z=0}$$

$$\frac{qd - q'd}{(\rho^2+d^2)^{3/2}} = \frac{q''d}{(\rho^2+d^2)^{3/2}} \Rightarrow \boxed{q - q' = q''}$$

Solving $\square$ eqns $\qquad q' = -\left(\dfrac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}\right)q \;;\quad q'' = \left(\dfrac{2\epsilon_2}{\epsilon_2+\epsilon_1}\right)q$

[1] cont'd An extension of this problem is to get the bound charge density $\sigma_b = P_{1z} - P_{2z}$ at the $z=0$ interface

[2.] In general, with azimuthal symmetry, the soln to Laplace Eqn is

$$V(r) = \sum\left[B_\ell r^\ell + C_\ell r^{-(\ell+1)}\right] P_\ell(\cos\theta)$$

If we use this for $r>a$ we have all $B_\ell = 0$ except since $-\vec{\nabla}V = E_0\hat{z}$ at $r=\infty$ we have $V = -E_0 z = -E_0 r\cos\theta$ as $r\to\infty$

$$V_>(r) = -E_0 r\cos\theta + \sum_\ell C_\ell r^{-(\ell+1)} P_\ell(\cos\theta)$$

For $r<a$ $\quad V_<(r) = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta)$ to avoid divergences at $r=0$

Now use bdy conditions on sphere surface

tangential E $\quad -\frac{1}{a}\frac{\partial V_>}{\partial\theta}\Big|_{r=a} = -\frac{1}{a}\,\partial V_</\partial\theta\Big|_{r=a}$

The coefficients of $P_\ell(\cos\theta)$ or $\partial/\partial\theta\, P_\ell(\cos\theta)$ must match term-by-term

$\ell = 1 \qquad A_1 = -E_0 + C_1/a^3$

$\ell \neq 1 \qquad A_\ell = C_\ell / a^{2\ell+1}$

Similarly for normal D $\quad -\epsilon\frac{\partial V_>}{\partial r}\Big|_{r=a} = -\epsilon_0\frac{\partial V_<}{\partial r}\Big|_{r=a}$

$\ell = 1 \qquad -\epsilon A_1 = +\epsilon_0\left(E_0 + 2C_1/a^3\right)$

$\ell \neq 1 \qquad -\epsilon\ell A_\ell = \epsilon_0(\ell+1)C_\ell/a^{2\ell+1}$

For $\ell\neq1$ the Eqns for A_ℓ, C_ℓ are homogeneous $\Rightarrow A_\ell = C_\ell = 0$

only A_1 and C_1 are nonvanishing

$$A_1 = \frac{-3}{\left(\frac{\epsilon}{\epsilon_0}+2\right)}E_0$$

$$C_1 = \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right)E_0$$

Inside sphere

$$V_< = \frac{-3}{\left(\frac{\epsilon}{\epsilon_0}+2\right)}E_0 \underbrace{r\cos\theta}_{z}$$

So $\vec{E} = -\nabla V_< = -3/(\epsilon/\epsilon_0 + 2)\,E_0\hat{z}$ uniform inside sphere

$\vec{D}$ and $\vec{P}$ are also constant within sphere $\quad \vec{P} = (\epsilon-\epsilon_0)\vec{E}$, $\quad \vec{D} = \epsilon\vec{E}$

[**3.**] Starting from the Maxwell Equations, show that the scalar and vector potentials obey separate wave equations when working in the Lorentz gauge.

-OR-

[**4.**] Derive the Greens function for the wave equation.

$\boxed{3.}$ a) $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$\vec{B} = \vec{\nabla}\times\vec{A}$ $\qquad$ $\frac{\partial \vec{E}}{\partial t} = \partial/\partial t\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t}\right)$

$$\nabla\times(\vec{\nabla}\times\vec{A}) = -\nabla^2 A + \vec{\nabla}(\vec{\nabla}\cdot A)$$

Thus $\quad -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \mu_0\vec{J} - 1/c^2\,\vec{\nabla}\,\partial\phi/\partial t - 1/c^2 \frac{\partial^2 A}{\partial t^2}$

$$\nabla^2\vec{A} - 1/c^2\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{A} + 1/c^2\,\partial\phi/\partial t)}_{\text{vanishes in Lorentz gauge}} = -\mu_0\vec{J}$$

$\Rightarrow$ $\vec{A}$ obeys wave eqn with $\mu_0\vec{J}$ as source

b) $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$

$\vec{\nabla}\cdot(-\vec{\nabla}\phi - \partial\vec{A}/\partial t) = \rho/\epsilon_0$

$\nabla^2\phi + \partial/\partial t(\vec{\nabla}\cdot\vec{A}) = -\rho/\epsilon_0$

$\vec{\nabla}\cdot\vec{A} = -1/c^2\,\partial\phi/\partial t$ in Lorentz gauge

$\Rightarrow$ ϕ obeys wave eqn with $+\rho/\epsilon_0$ as source

$\boxed{4.}$ We seek $G(\vec{r},t)$ obeying

$$(\nabla^2 - 1/c^2\,\partial^2/\partial t^2)\,G(\vec{r},t) = \delta^3(r)\delta(t)$$

Fourier transform to ω space

$$G(\vec{r},t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\tilde{G}(\vec{r},\omega)$$

and notice $\quad \delta(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}$

so that (defining $k=\omega/c$)

$$(\nabla^2 + k^2)\,\tilde{G}(\vec{r},\omega) = \delta^3(r)$$

At $\omega = k = 0$ we know the soln is $\tilde{G}(\vec{r},0) = 1/r\;1/4\pi$

If we look for a soln which only depends on $r = |\vec{r}|$ and not θ,ϕ even when $\omega\neq 0$ we can simplify the ∇^2 operator ...

$\boxed{4}$ Cont'd $\left[\underbrace{\frac{1}{r}\frac{d^2}{dr^2} r\tilde{G}}_{\nabla^2 \text{ if no } \theta,\phi \text{ dependence}} + k^2 \tilde{G} \right] = \delta^3(\vec{r})$

multiply by r and note $r\,\delta^3(\vec{r}) = 0$

$$\frac{d^2}{dr^2}(r\tilde{G}) = -k^2 r \tilde{G}$$

$$\Rightarrow \qquad r\tilde{G} = e^{\pm ikr}$$

Hence $\qquad \tilde{G}^{\pm}(r,\omega) = \frac{1}{r} e^{\pm ikr} \qquad k = \omega/c$

Can now go back to time domain

$$G^{\pm}(r,t) = \frac{1}{r}\int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, e^{\pm i\omega/c\, r}$$

$$= \frac{1}{r}\,\delta(\underbrace{t \mp r/c}_{\text{retarded time} \dots})$$

For completeness... you can decide which of $G^{\pm}$ obeys causality. We will want

$$\phi(rt) = \int d^3r' \int dt' \, \frac{\rho(r't')}{4\pi\epsilon_0} \underbrace{\frac{1}{r-r'}\,\delta\left(t - t' \mp \frac{|r-r'|}{c}\right)}_{G^{\pm}(r-r',\,t-t')}$$

we expect potential at time t to depend on charge density at $t' = \underbrace{t - \frac{|r-r'|}{c}}_{\underline{\text{earlier}} \text{ than } t}$

So it is G^{+} that obeys causality.