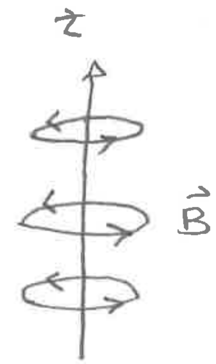
PHYSICS 200C, SPRING 2016
ELECTRICITY AND MAGNETISM

KEY

Midterm Exam

[1.] Consider the vector potential $\mathbf{A}(\mathbf{r}) = A_0\, e^{-(x^2+y^2)/a^2}\,\hat{\mathbf{z}}$, where A_0 and a are constants.
(a) Find and sketch the corresponding magnetic field. How is it similar/different from the field due to a long straight wire?
(b) Can this be a magneto*static* field? If yes, find the current distribution that would give rise to it, and if not, explain why not.
(c) Is the vector potential given in the Coulomb gauge?

a)
$$\vec{B} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & A_0 e^{-(x^2+y^2)/a^2} \end{vmatrix}$$



$$= \frac{2A_0}{a^2}\left(-y\hat{x} + x\hat{y}\right) e^{-(x^2+y^2)/a^2}$$

The vector structure is the same as a long straight wire.
The decay is much faster $-$ $e^{-(x^2+y^2)/a^2}$ vs $1/\sqrt{x^2+y^2}$.

b) We need to check if $\vec{\nabla}\cdot\vec{J} = \partial\rho/\partial t = 0$

We get $\vec{J}$ from $c/4\pi\ \vec{\nabla}\times\vec{B}$

$$\vec{\nabla}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -ye^{-(x^2+y^2)/a^2} & xe^{-(x^2+y^2)/a^2} & 0 \end{vmatrix}\frac{2A_0}{a^2}$$

$\hat{x}$ and $\hat{y}$ components vanish since $B_z = 0$ and B_x, B_y have no z dependence

$$= \frac{2A_0}{a^2}\hat{z}\left\{1 - \frac{2x^2}{a^2} + 1 - \frac{2y^2}{a^2}\right\} e^{-(x^2+y^2)/a^2}$$

$$\frac{4\pi}{c}\vec{J} = 4A_0/a^4 \left\{1 - x^2 - y^2\right\} e^{-(x^2+y^2)/a^2}\hat{z}$$

Clearly $\vec{\nabla}\cdot\vec{J} = 0$ because $J_x = J_y = 0$ and J_z has no z dependence.

c) For the same reason that $\vec{\nabla}\cdot\vec{J} = 0$ we also have $\vec{\nabla}\cdot\vec{A} = 0$:
$A_x = A_y = 0$ and $\partial A_z/\partial z = 0$, so, yes, we are in Coulomb gauge

[2.] In your homework, you solved for the Greens function of the diffusion equation,

$$\nabla^2 \vec{\mathbf{A}} = \mu \sigma \frac{\partial}{\partial t} \vec{\mathbf{A}} \tag{1}$$

Show that the vector potential $\vec{\mathbf{A}}$ does indeed obey the diffusion equation in a conducting medium where the current density $\vec{\mathbf{J}} = \sigma \vec{\mathbf{E}}$. You will need to use Maxwell's equations and the usual relations between the fields $\vec{\mathbf{E}}$, $\vec{\mathbf{B}}$ and the potentials Φ, $\vec{\mathbf{A}}$. You may assume there is no free charge so that $\Phi = 0$ and work in the Coulomb gauge $\nabla \cdot \vec{\mathbf{A}} = 0$.

If a current is flowing in a conductor, producing a vector potential $\vec{\mathbf{A}}$, and then the current is suddenly turned off, describe what Eq. 1 tells you qualitatively about how $\vec{\mathbf{A}}$ evolves. Assume the conductor fills all space.

We have $\vec{\nabla} \times \vec{B} = \mu \vec{J} = \mu \sigma \vec{E}$

in a conductor

Also $\vec{E} = -\vec{\nabla}\phi - \partial \vec{A}/\partial t = -\partial \vec{A}/\partial t$ in the absence of free charges

Finally $\vec{B} = \vec{\nabla} \times \vec{A}$ leading to

$$\underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{A})}_{\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}} = -\mu\sigma \partial \vec{A}/\partial t$$

$\nabla(\nabla\cdot\vec{A}) \to 0$

So we get Diffusion Eqn $\nabla^2 \vec{A} = \mu\sigma \partial \vec{A}/\partial t$

[3.] The Greens function for the one dimensional diffusion equation obeys

$$\left(D\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right)\mathcal{G}(x,t) = \delta(x)\delta(t). \tag{2}$$

Write

$$\mathcal{G}(x,t) = \int dk\, d\omega\, e^{i(kx-wt)}\,\mathcal{G}(k,\omega) \tag{3}$$

and find a formula for $\mathcal{G}(k,\omega)$. Insert this into Eq. 3 and do the ω integration. How does causality appear from the mathematics? If you have time, do the k integration as well, or if not, state the general procedure for doing it.

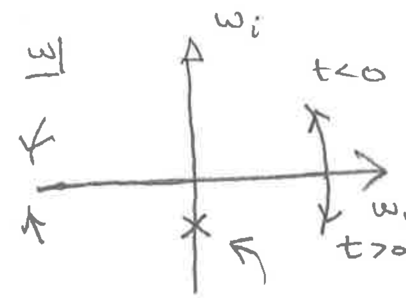
Plugging (3) into (2) and identify the coefficients of $G(k,w)$ on each side

$$(-Dk^2 + i\omega)\,G(k\,\omega) = 1/(2\pi)^2$$

Here we used $\delta(x) = \frac{1}{2\pi}\int dx\, e^{ikx}$

and similarly for $\delta(t)$.

$$G(k\;\omega) = \frac{1}{(2\pi)^2}\,\frac{1}{i\omega - Dk^2}$$



Inserting this into (3) to get $G(x\,t)$

$$G(x,t) = \int \frac{dk\,d\omega}{(2\pi)^2}\, e^{i(kx-\omega t)}\; 1/i\omega - Dk^2 \quad \leftarrow \text{pole at } \omega = -iDk^2$$

The ω integration uses $e^{-i\omega t} = e^{-i\omega_r t}\, e^{\omega_i t}$

For $t<0$ we need to close contour with $\omega_i > 0$ to get $e^{\omega_i t} \to 0$

We do not enclose the pole and $G(x\,t) = 0$.

For $t>0$ we need to close contour with $\omega_i < 0$ (lower half plane) and we do pick up the pole. We get $2\pi i \times \{\text{Residue}\}$

$$G(x\,t) = -i\int \frac{dk}{2\pi}\, e^{ikx}\, e^{-i(-iDk^2)t}\,\theta(t) = \frac{-i}{2\pi}\int dk\, e^{-Dk^2 t + ikx}\,\theta(t)$$

The k integral is done via completing the square

$$-Dk^2 t + ikx = -Dt\left(k - \frac{ix}{2Dt}\right)^2 - x^2/4Dt$$

$$G(x\,t) = \frac{-i}{2\pi}\, e^{-x^2/4Dt}\,\theta(t)\,\sqrt{\frac{\pi}{Dt}}$$

3 (cont'd)

We expect $G(x\,t)$ is normalized to 1 because $G(x,t)$ is supposed to represent the sol'n to eqn with a $\delta(x)$ source, and the diffusion eqn preserves the particle number$^{(*)}$. Let's check

$$\int_{-\infty}^{\infty} dx\, \frac{1}{2\pi}\sqrt{\frac{\pi}{Dt}}\, e^{-x^2/4Dt} = \frac{1}{2\pi}\sqrt{\frac{\pi}{Dt}}\,\sqrt{4\pi Dt} = 1 \ \checkmark$$

(*) what this means is that if $\psi(x,t)$ is a sol'n of the diffusion eqn

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \psi(x\,t)\,dx = \int_{-\infty}^{\infty} D\frac{\partial^2}{\partial x^2}\psi(x\,t)\,dx$$

integrate by parts

$$= \underbrace{D\frac{\partial \psi}{\partial x}\Big|_{-\infty}^{\infty}}_{\phi} - \int_{-\infty}^{\infty} \underbrace{\frac{\partial D}{\partial x}}_{\phi}\frac{\partial \psi}{\partial x}\,dx$$

assuming ψ and $\frac{\partial \psi}{\partial x}$ vanish at $\pm\infty$

so $\quad \partial/\partial t \int_{-\infty}^{\infty} \psi(x,t)\,dx = 0$