L1

Electromagnetic Waves in a Laser Beam

Plane waves are interesting but a bit bizarre

Since they assume indefinite extent in space, lets

look at EM fields in laser beam – a problem

of intrinsic interest but which also allows a "more

physical" situation where spatial extent finite

This analysis follows a fairly recent experimental paper (1994)

We will proceed by writing

Although extent is finite we will assume variation is on scale $\gg \lambda$

$$\vec{E}(r,t) = \vec{e}(r)\, e^{i(kz-\omega t)}$$

$$\vec{B}(r,t) = \vec{b}(r)\, e^{i(kz-\omega t)}$$

Modulated by additional function

usual plane wave

A bit like "Common Approximation" strategy (in QM eg) general strategy soln to problem + perturbation is old soln $\times$ new function

$k = \omega/c$

$e \parallel \hat{x}$ "mostly"

$b \parallel \hat{y}$ "mostly"

we will see what this means later

$$e_x(r) = \chi(r)$$

χ is slowly varying on scale $1/k \sim \lambda$

$$\frac{d\chi}{dz} \ll k\chi$$

L2

Wave Eqn for E_x can be written

$$E_x(r,t) = \chi(r)\, e^{i(kz-\omega t)}$$

no x, y dependence

$$\frac{\partial^2}{\partial x^2} E_x(r,t) = \left(\frac{\partial^2}{\partial x^2}\chi\right) e^{i(kz-\omega t)}$$

$$\frac{\partial^2}{\partial y^2} E_x(r,t) = \left(\frac{\partial^2 \chi}{\partial y^2}\right) e^{i(kz-\omega t)}$$

$$\frac{\partial^2}{\partial z^2} E_x(r,t) = \frac{\partial}{\partial z}\left[\frac{\partial \chi}{\partial z} e^{i(kz-\omega t)} + ik\chi\, e^{i(kz-\omega t)}\right]$$

$$= \left[\frac{\partial^2 \chi}{\partial z^2} + 2ik\frac{\partial \chi}{\partial z} - k^2\chi\right] e^{i(kz-\omega t)}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} E_x(r,t) = -\frac{\omega^2}{c^2}\chi\, e^{i(kz-\omega t)}$$

match up

$$\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} + 2ik\frac{\partial \chi}{\partial z} = 0$$

Assumption is that $\frac{\partial \chi}{\partial z} \ll k\chi$ $\Big\{$ χ varies slowly on length $1/k$

So $\frac{\partial^2 \chi}{\partial z^2}$ is neglectable

Cannot throw away $2ik\frac{\partial \chi}{\partial z}$

or else get Laplace eqn in 2D soln is $\chi = $ const.

$\Rightarrow$ usual plane wave soln.

We are looking for a solution localized in x, y

so make an ansatz

$$\chi(\vec{r}) = h(z)\, e^{-(x^2+y^2)/2g(z)}$$

Why? Some simple function which decays away from z axis. But allow for width and magnitude to vary with z.

$$\frac{\partial \chi}{\partial x} = h(z) \left(\frac{-2x}{2g(z)}\right) e^{-(x^2+y^2)/2g(z)}$$

$$\frac{\partial^2 \chi}{\partial x^2} = \frac{-h(z)}{g(z)} e^{-(x^2+y^2)/2g(z)} + \frac{h(z)\, x^2}{g^2(z)} e^{-(x^2+y^2)/2g(z)}$$

So $$\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} = h(z) \left\{ \frac{x^2+y^2}{g^2(z)} - \frac{2}{g(z)} \right\} e^{-(x^2+y^2)/2g(z)}$$

$2ik$

$$\frac{\partial \chi}{\partial z} = \left[dh/dz + h \frac{(x^2+y^2)}{2g^2(z)} \frac{\partial g}{\partial z} \right] e^{-(x^2+y^2)/2g(z)}$$

The top expression $+2ik$ times bottom must vanish.

A sufficient conditions is

$$\frac{dh}{dz} = \frac{-i}{k}\frac{h}{g} \quad \text{and} \quad h \frac{dg}{\partial z} \frac{1}{g^2} = \frac{i}{k}\frac{h}{g^2}$$

We do this by matching the similar looking terms and in particular matching terms with x, y dependence and without

L4

$$dg/dz = i/k \implies g(z) = \frac{i}{k}(z - iz_0)$$

$$\frac{dh}{dz} = -\frac{i}{k}\frac{h}{g} = -\frac{h}{(z-iz_0)}$$

z_0 can be chosen real since imag part of z_0 just shifts zero of z.

$$\frac{dh}{h} = -dz/(z-iz_0)$$

$$\ln h = -\ln(z-iz_0) + C$$

$$h = \frac{\alpha_0}{z-iz_0}$$

$$\chi(\vec{r}) = \chi(x,y,z) = \frac{\alpha_0}{z-iz_0}\exp\left\{\frac{ik}{2}\left(\frac{x^2+y^2}{z-iz_0}\right)\right\}$$

$$= \frac{\alpha}{z-iz_0}\exp\left\{-\frac{x^2+y^2}{2w^2(z)} + i\frac{k}{2}\frac{x^2+y^2}{R(z)}\right\}$$

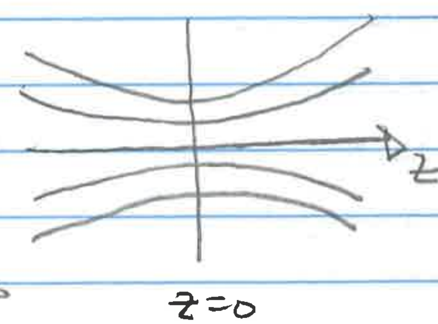
$$\frac{1}{z-iz_0} = \frac{z+iz_0}{z^2+z_0^2}$$

$$w(z) = \left(\frac{z^2+z_0^2}{kz_0}\right)^{1/2} \qquad R(z) = \frac{z^2+z_0^2}{z}$$

This is transverse width of beam as it propagates

called the beam waist

$w(z)$ is smallest at ~~a~~ $z=0$ where $w(0) = \sqrt{\frac{z_0}{k}}$



L5A

Plane EM wave $\vec{E}(r) = \vec{E}_0(r)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

Suppose we write

collect all k dependence here

$$\vec{E}(\vec{r}) = \vec{E}_0(\vec{r})\, e^{i(k\psi(r) - \omega t)}$$

$$\vec{B}(\vec{r}) = \vec{B}_0(\vec{r})\, e^{i(k\psi(r) - \omega t)}$$

$\psi(r)$ is called the "eikonal"

$$\vec{\nabla}\cdot\vec{B} = \left(\vec{\nabla}\cdot\vec{B}_0 + ik\vec{B}_0\cdot\nabla\psi\right) e^{i(k\psi(r) - \omega t)}$$

Assume variation of fields is on scale $\gg \lambda \Rightarrow \underbrace{\frac{\partial}{\partial x} \ll k}_{\text{"geometrical optics"}}$

So drop first term in $\vec{\nabla}\cdot\vec{B}$.

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla}\psi\cdot\vec{B}_0 = 0$$

$$\vec{\nabla}\cdot\vec{E} = 0 \quad \Rightarrow \quad \nabla\psi\cdot\vec{E}_0 = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \quad \Rightarrow \quad \vec{\nabla}\psi\times\vec{E}_0 - \vec{B}_0 = 0$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \qquad \vec{\nabla}\psi\times\vec{B}_0 + \vec{E}_0 = 0$$

$*$ $\vec{E}_0$, $\vec{B}_0$ and $\nabla\psi$ are mutually $\perp$

also $$\nabla\psi\times\overbrace{(\nabla\psi\times E_0)}^{\vec{B}_0} + \vec{E}_0 = 0$$

$$\vec{\nabla}\psi(\vec{\nabla\psi}\cdot\vec{E}_0) - \vec{E}_0|\nabla\psi|^2 + \vec{E}_0 = 0$$

$$\Rightarrow |\nabla\psi|^2 = 1$$

L5B

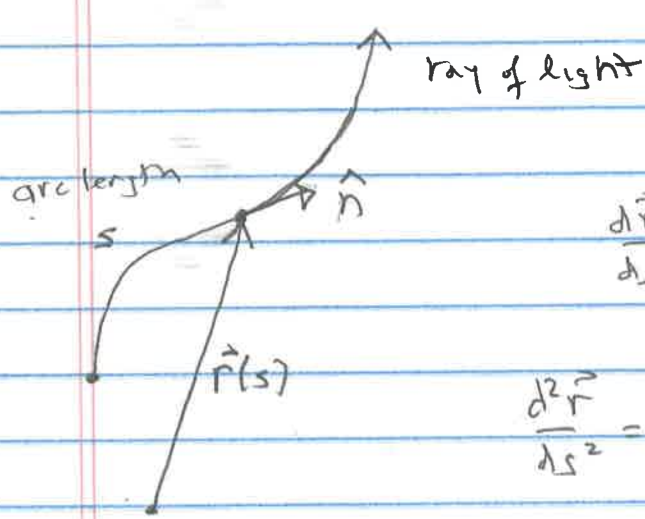
$|\nabla\psi|^2 = 1$ Basic Eqn of geometrical optics "eikonal Eqn"

$\nabla\psi = \hat{n}$ ← direction of energy flow $\vec{S}$

Energy flows $\perp$ to surfaces of ψ = constant

ψ = const surfaces : "wave fronts"

lines of Poynting vector $\vec{S}$: "rays"



$$\frac{d\vec{r}}{ds} = \frac{\vec{r}(s+ds) - \vec{r}(s)}{ds} = \hat{n} = \vec{\nabla}\psi[\vec{r}(s)]$$

$$\frac{d^2\vec{r}}{ds^2} = \frac{d\vec{r}}{ds}\cdot\frac{d}{d\vec{r}}\vec{\nabla}\psi = (\underbrace{\vec{\nabla}\psi}_{\frac{dr}{ds}}\cdot\vec{\nabla})\vec{\nabla}\psi$$

Vector identity $\nabla(\vec{u}\cdot\vec{u}) = 2(\vec{u}\cdot\vec{\nabla})\vec{u} + 2\vec{u}\times(\underbrace{\vec{\nabla}\times\vec{u}}_{\nabla\psi \text{ vanishes}})$

Use $\vec{u} = \nabla\psi$

$$\frac{d^2\vec{r}}{ds^2} = \frac{1}{2}\nabla(\underbrace{\nabla\psi\cdot\nabla\psi}_{1}) = 0$$

L5C

$\hat{n} = d\vec{r}/ds$ = Constant

rays are straight lines !

MATH NOTES $\vec{\nabla}(\vec{u}\cdot\vec{u})$ ith component

$$\frac{\partial}{\partial x_i} u_j u_j = 2 u_j \frac{\partial}{\partial x_i} u_j$$

$2(\vec{u}\cdot\vec{\nabla})\vec{u} + 2\vec{u}\times(\vec{\nabla}\times\vec{u})$ ith component

$$2 u_j \frac{\partial}{\partial x_j} u_i + 2 \epsilon_{ijk} u_j (\vec{\nabla}\times\vec{u})_k$$

$$\downarrow$$

$$\epsilon_{klm} \frac{\partial}{\partial x_l} u_m$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{lj}$$

$$\cancel{2 u_j \frac{\partial}{\partial x_j} u_i} + 2 u_j \frac{\partial}{\partial x_i} u_j \cancel{- 2 u_j \frac{\partial}{\partial x_j} u_i}$$

L5D

Let us examine the "wave fronts". These are

surfaces of constant spatial phase ψ of $\vec{E}$

$$\vec{E} = \vec{e}(r)\, e^{i(kz-\omega t)} = E(\vec{r})\, e^{i(k\psi-\omega t)}$$

absorb k dependent phases in $\vec{e}$ into $\vec{E}$

$$\propto \frac{}{z - i z_0} \exp\left\{-\frac{x^2+y^2}{2w^2(z)} + i\frac{k}{2}\frac{x^2+y^2}{R(z)}\right\}$$

ψ is called the "eikonal"

$$w^2 = \frac{z^2+z_0^2}{kz_0} \qquad R = \frac{z^2+z_0^2}{z}$$

so spatial phase is $\psi \equiv z + \frac{x^2+y^2}{2R(z)}$

The significance of ψ = const is that $\vec{E}$ field has

same phase at all points on this surface

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\frac{\partial E_x}{\partial x} e^{i(k\psi-\omega t)} + ikE_x e^{i(k\psi-\omega t)}\frac{\partial \psi}{\partial x}$$

neglect this term since E varying slowly compared to $\vec{k}$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = (\vec{\nabla}\psi\cdot\vec{E})ik$$

LSE

Similarly for $\vec{B}$

$\vec{E}, \vec{B}$ are $\perp$ to $\vec{\nabla}\psi$

surfaces of constant phase

aka wave fronts.