RW1

Finite string

0 L

$$y(x,t) = \sin\frac{n\pi x}{L}\left[a_n\cos\frac{n\pi v t}{L} + b_n\sin\frac{n\pi v t}{L}\right]$$

Determined by $y(x,t=0)$ — by $\frac{\partial y}{\partial t}(x,t=0)$

Green's function

$(x,t) = \int dx'\, G(x,x',t)\, y(x,0)$

"standing wave"

$\sin\frac{n\pi x}{L}\cos\frac{n\pi v t}{L}$

$t=0$

$t=\frac{L}{2n\pi v}$

$t=\frac{L}{n\pi v}$

$\sin A\cos B$

$$= \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$= \frac{1}{2}\left[\sin\frac{n\pi}{L}(x+vt) + \sin\frac{n\pi}{L}(x-vt)\right]$$

"traveling waves"

General soln to wave eqn

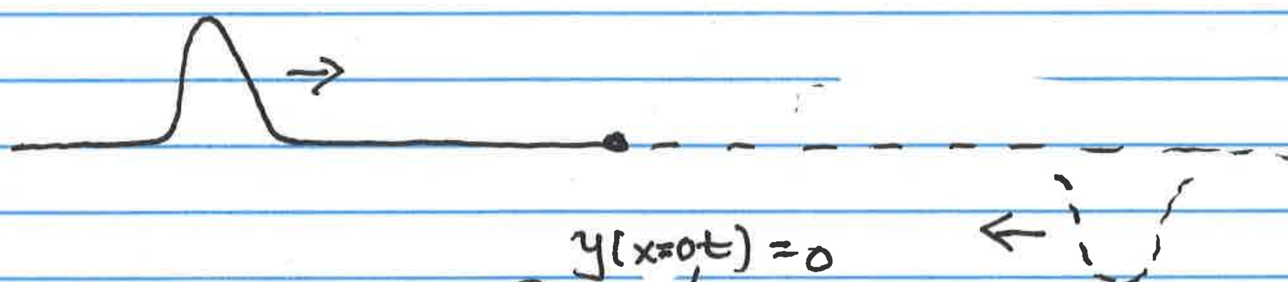
$f(x-vt)$ $f(x+vt)$

any function!

RW2

This is a remarkable feature of wave eqn associated with symmetry $x \leftrightarrow t$. Not true of Sch. Eqn or diffusion eqn ...

Actually we alluded to this



$y(x=0, t) = 0$

semi infinite string

solve wave eqn with boundary condition

initial condition

$$y(x, t=0) = e^{-(x+20)^2}$$

ie gaussian centered @ $x=-20$

$$y(x,t) = e^{-[x+20-vt]^2} - e^{-[x-20+vt]^2}$$

image pulse

Any thing funny here?!

why not $e^{-[x+20+vt]^2}$?

RW 3

Incomplete information (initial conditions)!

Infinite string

$$y(x, t=0) = e^{-x^2}$$

$$y(x,t) = \int_{-\infty}^{\infty} e^{ikx}\left[a(k)e^{ikvt} + b(k)e^{-ikvt}\right] dk$$

$$y(x,t=0) = \int_{-\infty}^{\infty} e^{ikx}\left[a(k) + b(k)\right] dk$$

$$a(k) + b(k) = \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-ikx} \underbrace{y(x,t=0)}_{e^{-x^2}}$$

Complete the square etc

$$\frac{\partial}{\partial t} y(x,t) = \int_{-\infty}^{\infty} e^{ikx}\, ikv \left[a(k)e^{ikvt} - b(k)e^{-ikvt}\right] dk$$

$$\frac{\partial}{\partial t} y(x,t=0) = \int_{-\infty}^{\infty} e^{ikx}\, ikv \left[a(k) - b(k)\right] dk$$

$$ikv\left[a(k) - b(k)\right] = \int_{-\infty}^{\infty} e^{-ikx} \frac{\partial y(x,t=0)}{\partial t} \frac{dx}{2\pi}$$

Electromagnet wave

We turn now to a particular type of radiation – electromagnetic waves.

We have seen $\vec{E}$ and $\vec{B}$, like $\vec{A}$ and ϕ obey the wave eqn. In a vacuum $\rho = J = 0$

$$\nabla^2 B - \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} = 0 \qquad \mu_0 \epsilon_0 = 1/c^2$$

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

We have also discussed the wave eqn and its solutions

In one dimension $\partial^2 f/\partial x^2 - 1/c^2 \, \partial^2 f/\partial t^2$

has solutions $f(x,t) = f_1(x-ct) + f_2(x+ct)$

right moving   left moving

which are further restricted by boundary conditions (eg wave on a string tied down at ends)

$E_x(x,y,z,t)$
$E_y(x,y,z,t)$
$E_z(x,y,z,t)$

$\vec{E},\vec{B}$ Waves are much richer

(1) 3D
(2) $\vec{E},\vec{B}$ are coupled
(3) $\vec{E},\vec{B}$ vectors

vibrational waves in solids "phonons" are also richer for same reason

L, T different from E, B

$\vec{R}_n(x,y,z,t)$
$X_n(x,y,z,t)$
Y_n "
Z_n
(discrete) position of nucleus n

Indeed in the coulomb gauge where $\vec{\nabla}\cdot\vec{A}=0$

the relations $\vec{E} = -\nabla\phi - \frac{1}{c}\partial\vec{A}/\partial t$ and $\vec{\nabla}\cdot\vec{E}=0$

imply $\nabla^2\phi = 0$ which has solution $\phi = 0$

Thus $\vec{B} = \vec{\nabla}\times\vec{A}$ $\quad \vec{E} = -\frac{1}{c}\,\partial\vec{A}/\partial t$

so $\vec{E}$ and $\vec{B}$ depend on single vector field $\vec{A}$. This

Q

is the general approach. But for plane waves we can proceed more simply.

fields vary only in 1 direction. Call this $\hat{z}$

consider $\hat{z}$ component of $\nabla^2\vec{E} - \frac{1}{c^2}\partial^2\vec{E}/\partial t^2 = 0$

$$\partial^2 E_z/\partial z^2 - \frac{1}{c^2}\,\partial^2 E_z/\partial t = 0.$$

$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \partial E_z/\partial z = 0$ since plane wave varying only in z direction.

Thus $\partial^2 E_z/\partial t^2 = 0$ $\quad E_z = a + bt$

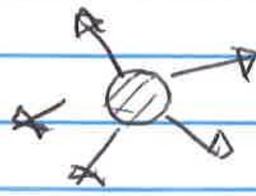
trivial

unphysical (grows with time)

(Q) Why does $\vec{\nabla}\cdot\vec{E} = 0$ and $E_\alpha(x,y,z) = E_\alpha(z)$

imply $E_z = 0$?

Divergence

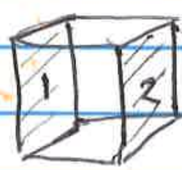


$\Rightarrow$ flow through surface

Vector field with $E_\alpha(x,y,z) = E_\alpha(z)$

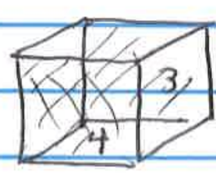
in given plane $\perp$ $\hat{z}$

vectors all identical



Cannot be any net flow through sides of box (1, 2)

or pair (3, 4)



Only possible flux is Top + Bottom surfaces

$$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \int_{\text{TOP}} E_z\, dA = \int_{\text{BOTTOM}} E_z\, dA$$

$$\Rightarrow E_z = \text{constant}$$

EMW-3

We conclude $E_z = 0$. Similarly $B_z = 0$

The x and y components of $\vec{E}$ and $\vec{B}$ also obey a 1-d wave eqn, we will consider a right moving solution (The Ampere Eqn tells us $\vec{B}$ must be right-moving if $\vec{E}$ is)

$$\left.\begin{aligned} \vec{E} &= \vec{E}(z-ct) \\ \vec{B} &= \vec{B}(z-ct) \end{aligned}\right\} \text{only } x,y \text{ components are nonzero}$$

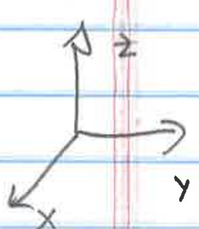
Faraday $\quad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\, \partial \vec{B}/\partial t$

no x,y dependence

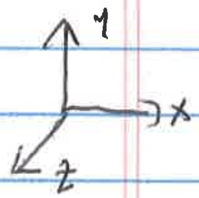
$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \boxed{\partial/\partial x \quad \partial/\partial y} & & \partial/\partial z \\ E_x & E_y & 0 \end{vmatrix} \rightarrow -\hat{x}\frac{\partial E_y}{\partial z} + \hat{y}\frac{\partial E_x}{\partial z}$$

$E_z = 0$

$$= -\frac{1}{c}\left(\hat{x}\frac{\partial B_x}{\partial t} + \hat{y}\, \partial B_y/\partial t\right)$$



$$\hat{z} \times (\hat{x} V_x + \hat{y} V_y) = \hat{y} V_x - \hat{x} V_y$$



$$\left(-\hat{x}\,\partial E_y/\partial z + \hat{y}\,\partial E_x/\partial z\right) = -1/c\left(\hat{x}\,\partial B_x/\partial t + \hat{y}\,\partial B_y/\partial t\right)$$

$$\hat{z}\times\left(\hat{x}\,\partial E_x/\partial z + \hat{y}\,\partial E_y/\partial z\right) = -1/c\left(\hat{x}\,\partial B_x/\partial t + \hat{y}\,\partial B_y/\partial t\right)$$

$u = x - ct$

$f(x-ct)$

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}$

$\frac{\partial f}{\partial t} = -c\,\frac{\partial f}{\partial u}$

$$\hat{z}\times\frac{\partial \vec{E}}{\partial u} = -1/c\;\partial\vec{B}/\partial u\;(-c)$$

$$\hat{z}\times\frac{\partial \vec{E}}{\partial u} = +\frac{\partial \vec{B}}{\partial u}$$

Integrating $\hat{z}\times\vec{E} = \vec{B}$

or $\hat{z}\times(\hat{z}\times\vec{E}) = \hat{z}\times\vec{B}$

$\hat{z}(\hat{z}\cdot\vec{E})^{0} - \vec{E}(\underbrace{\hat{z}\cdot\hat{z}}_{1})$

Similar analysis of $\vec{\nabla}\times\vec{B} = 1/c\;\partial\vec{E}/\partial t$

yields $\hat{z}\times\vec{B} = -\vec{E}$

Obviously the choice of $\hat{z}$ as a variable was arbitrary, For a general direction $\hat{n}$

$\vec{E}\perp\hat{n}$ $\quad\vec{B}\perp\hat{n}$ $\quad\vec{B} = \hat{n}\times\vec{E}$

In SI units $\vec{B} = 1/c\;\hat{n}\times\vec{E}$

EMW-5

The Poynting vector

speed

Energy density $\frac{1}{8\pi}|E|^2$ times 2 for $\vec{B}$ field

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}|E|^2\hat{n}$$

In SI units $\quad \vec{S} = c\,\epsilon_0 |E|^2\,\hat{n}$

Speed $\quad$ Energy density $\frac{1}{2}\epsilon_0|E|^2$

(times 2 for $\vec{B}$ field)

Momentum density $\quad \vec{g} = \frac{1}{4\pi c}|E|^2\,\hat{n}$

Energy = c (momentum)

SI $\quad \vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \qquad c^2 = \frac{1}{\mu_0\epsilon_0}$

$$= \epsilon_0 c^2\,\vec{E}\times\vec{B}$$

$$\vec{B} = \frac{1}{c}\hat{n}\times\vec{E}$$

$$= c\,\epsilon_0 |E|^2\,\hat{n}$$

$\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp}$ $\parallel$ or $\perp$ to $\vec{k}$

$\vec{k}\cdot\vec{r} = \vec{k}\cdot\vec{r}_{\parallel}$ $\vec{E}, \vec{B}$ indep of $\vec{r}_{\perp}$

Let us consider a particular form for $\vec{E}$

which is "monochromatic" ← single frequency

$$\vec{E}(\vec{r}t) = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\vec{k} = \omega/_c\, \hat{n} \qquad \vec{E}_0 \perp \hat{n}$$

$$\vec{B}(rt) = \vec{B}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$\vec{E}_0$ and $\vec{B}_0$ are complex vectors

$$\vec{B}_0 = \hat{k} \times \vec{E}_0 \qquad \frac{1}{c}\hat{k}\times\vec{E}$$

so identify $\hat{k} = \hat{n}$

Take Real part to get physical fields

$$\hat{k} = \hat{n}$$

It turns out that with an appropriate choice of phase angle α one can write

$$\vec{E}_0 = (\vec{E}_{0r} + i\vec{E}_{0i}) \qquad E = E_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t - \alpha)}$$

with $\vec{E}_{0r} \perp \vec{E}_{0i}$

This is done by starting with $\vec{E} = (\vec{\mathcal{E}}_{0r} + i\vec{\mathcal{E}}_{0i})e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

not orthogonal

and getting $\alpha = -\frac{1}{2}\tan^{-1}\dfrac{2\vec{\mathcal{E}}_{0r}\cdot\vec{\mathcal{E}}_{0i}}{\mathcal{E}_{0r}^2 - \mathcal{E}_{0i}^2}$

EMW7

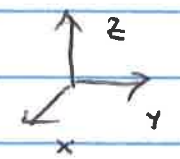
If $\vec{k} \parallel \hat{z}$

Choosing $\vec{E}_{0r}$ along $\hat{x}$ axis $\vec{E}_{0i}$ will be along $\hat{y}$ axis

$$\vec{E}(\vec{r}t) = \left(E_{0x}\hat{x} + iE_{0y}\hat{y}\right) e^{i(\vec{k}\cdot\vec{r} - \omega t - \alpha)}$$

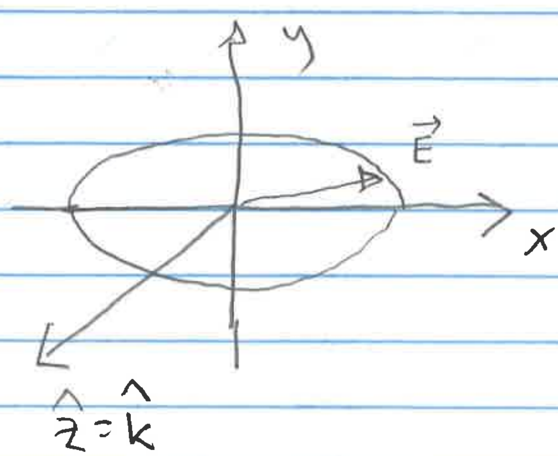
Most general form of monochromatic real electric field

$$\left\{\begin{aligned} E_x(\vec{r}t) &= E_{0x}\cos(\vec{k}\cdot\vec{r} - \omega t - \alpha) \\ E_y(\vec{r}t) &= -E_{0y}\sin(\vec{k}\cdot\vec{r} - \omega t - \alpha) \\ B_x(\vec{r}t) &= -E_y(rt) \qquad \vec{B} = \hat{z}\times\vec{E} \\ B_y(rt) &= E_x(rt) \end{aligned}\right.$$



$$\frac{E_x^2(\vec{r}t)}{E_{0x}^2} + \frac{E_y^2(rt)}{E_{0y}^2} = 1$$ ← Eqn of an ellipse

At a fixed point $\vec{r}$ in space, the tip of Electric field vector describes an ellipse in xy plane



EMW 8

SPECIAL CASES

$E_{0x} = E_{0y}$ Right circularly polarized

$E_{0x} = -E_{0y}$ Left " "

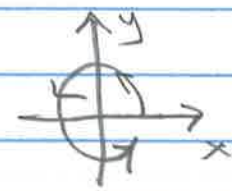
"Circle" is obvious : semimajor axes same

To see left/right consider shifting time

so that $\vec{k} \cdot \vec{r} - \alpha = 0$

$E_x = E_{0x} \cos(-\omega t) = E_{0x} \cos \omega t$

$E_y = -E_{0y} \sin(-\omega t) = E_{0y} \sin \omega t$



Counter clockwise (Right hand rule)

Left circularly polarized $\rightarrow$ clockwise "left"

RCP $\vec{E}(rt) = E_{0x} (\hat{x} + i\hat{y})\, e^{i(\vec{k} \cdot \vec{r} - \omega t - \alpha)}$

LCP $\vec{E}(\vec{r}t) = E_{0x} (\hat{x} - i\hat{y})\, e^{i(\vec{k} \cdot \vec{r} - \omega t - \alpha)}$

EMW 9

Linearly polarized

$E_{0x} = 0$ or $E_{0y} = 0$