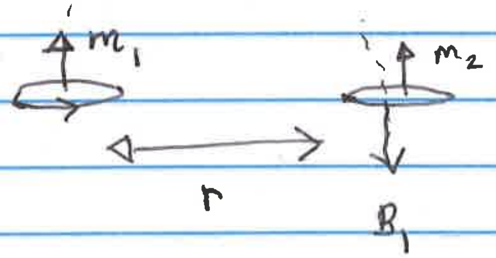
E/M Energy Scales

$$U_{el} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} = (9\cdot 10^9)\frac{(1.6\cdot 10^{-19})^2}{10^{-10}} = 2.3\cdot 10^{-19}\,J = 1.44\,eV$$

expected since calc is sort of like K atom

U_{mag}

$= m_2 \cdot B_1$



$$m_1, m_2 \sim \mu_B = \frac{e\hbar}{2m_e} = 9.27\cdot 10^{-24}\,\frac{J}{T}$$

$$B_1 = \frac{\mu_0}{4\pi r^3}\left\{3\frac{\vec{m}_1\cdot\vec{r}}{r^2} - \vec{m}_1\right\}$$

$$\frac{4\pi\cdot 10^{-7}}{4\pi(10^{-10})^3}(9.27\cdot 10^{-24})^2$$

$$= 8.6\cdot 10^{-24}\,J = 5.4\cdot 10^{-5}\,eV$$

$\sim 1\,°K$

→ origin of magnetism?

Forces $F_{el} = \frac{1}{4\pi\epsilon_0} e^2/r^2 = 9\cdot 10^9 \frac{(1.6\cdot 10^{-19})^2}{(10^{-10})^2} = 2.3\cdot 10^{-9}\,N$

This is large $a = F_{el}/m = 2.3\cdot 10^{-9}/9.11\cdot 10^{-31}$

$= 2.5\cdot 10^{21}\,m/s^2$

$$F_{mag} = e\vec{v}\times\vec{B} = (1.6\cdot 10^{-19})v\,\underbrace{\frac{4\pi\cdot 10^{-7}}{4\pi(10^{-10})^3}}_{\frac{\mu_0}{4\pi r^3}}\,\underbrace{9.27\cdot 10^{-24}}_{\mu_B}$$

$F_{mag} = 1.6\cdot 10^{-19}\,v\,(1)$

$F_{el} = 2.3\cdot 10^{-9}$

0.927 T !

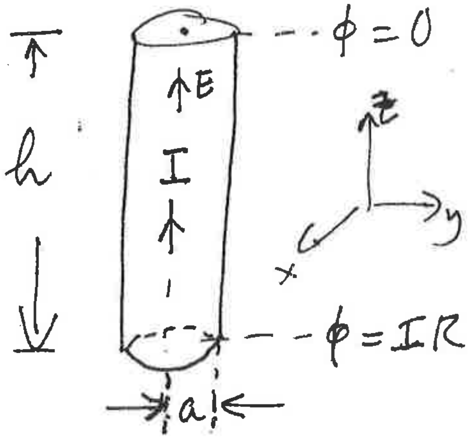
Lecture

March 19, 2014

Electricity & Magnetism (414-1)

Solutions to Set 9

1&2 Poynting vector for current carrying fat wire.



If the wire has resistance R, the potential drop is IR as shown and

$$\vec{E} = -\vec{\nabla}\phi = \frac{IR}{h}\hat{z} \quad , \qquad (1)$$

everywhere inside the wire and on its surface.

Next, let us find $\vec{B}$ right on the surface of the wire. Ampere's law gives

$$\vec{B} = (2I/ac)\,\hat{e}_\varphi \quad \text{(Gaussian)} \, . \qquad (2)$$

$$\therefore \; \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{I^2R}{2\pi a h}(-\hat{e}_r) \, . \qquad (3)$$

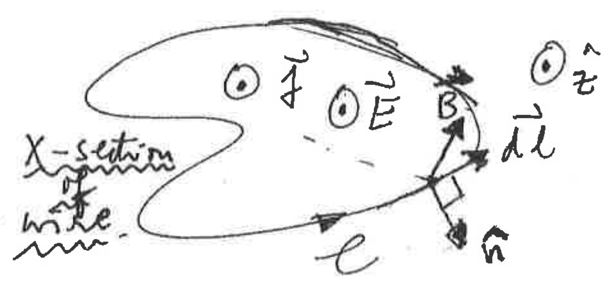
Integrating over the entire surface area of the segment of wire shown,

$$\text{Energy flowing in per unit time} = \frac{I^2R}{2\pi a h}\,2\pi a h = I^2R \, . \qquad (4)$$

$$\text{Energy lost per unit time as heat} = I^2R \quad \left(\text{from 100 Level Physics}\right). \qquad (5)$$

The surprise is that this energy is supplied not along the wire, but from the sides.

- If the wire has a non circular cross-section, denote its boundary by $\mathcal{C}$.



Again consider a segment of length h along the $\hat{z}$ axis. Then, as before

$$\vec{E} = (IR/h)\,\hat{z}. \tag{6}$$

Once again $\vec{B}$ is in the xy plane, but we cannot say that it is tangent to the surface of the wire. In other words

$$\vec{B} \not\parallel d\ell \;;\quad \vec{B} \not\perp \hat{n}. \tag{7}$$

Note that

$$(\hat{z} \times \hat{n}) \parallel \vec{d\ell}. \tag{8}$$

$\hat{n}$ is the outward normal to the wire, lies in xy plane.

$$\text{Energy inflow per unit time} = -h \oint_{C} (\vec{S}\cdot\hat{n})\, \underbrace{d\ell}_{\substack{\text{Note scalar, i.e}\\ \text{magnitude of length element}}} \tag{9}$$

$$-\vec{S}\cdot\hat{n}\, d\ell = -\frac{c}{4\pi}(\vec{E}\times\vec{B})\cdot\hat{n}\, d\ell \tag{10a}$$

$$= \frac{c}{4\pi}\vec{B}\cdot\underbrace{(\vec{E}\times\hat{n})\,d\ell}_{=\text{ vector }\parallel \vec{d\ell}} = E_z\, \vec{d\ell} \tag{10b}$$

$$\circledast \therefore \text{RHS (9)} = \frac{c}{4\pi} h\, E_z \oint_{C} \vec{B}\cdot\vec{d\ell} \underset{\wedge}{\overset{(c/4\pi)}{=}} h\, E_z\left(\frac{4\pi}{c} I\right) \text{ by Ampere's law}$$

$$= I^2 R. \tag{11}$$

And this is exactly the power dissipated once again.
The key point of this problem is to recognize that we want the form $\oint \vec{B}\cdot\vec{dl}$ in $\circledast$, since this is the one surely known thing.

Lecture ?

Spring 2013-14 Electricity and Magnetism (414–2) Apr. 7, 2014

Mini-Test 7

Give reasons for all answers, and show enough work that I can follow your reasoning.

1 (15 points). A metal box in the shape of a cube of side a acts as a cavity resonator. The EM fields in one of the resonant modes are given by (in the Gaussian system)

$$\mathbf{E} = E_0 \cos kx \cos ky \cos \omega t\, \hat{\mathbf{z}}, \qquad \mathbf{B} = \frac{E_0}{\sqrt{2}} (\cos kx \sin ky\, \hat{\mathbf{x}} - \sin kx \cos ky\, \hat{\mathbf{y}}) \sin \omega t,$$

where $k = \pi/a$, and $\omega = \sqrt{2}kc$. The origin is taken at the center of the cube.

(a) Show that energy is locally conserved, i.e., the continuity equation for energy is obeyed.

(b) Sketch the energy flow at an instant when it is not zero everywhere. Mark and label your sketch informatively.

(a) Energy density $= u = \frac{1}{8\pi}(E^2 + B^2)$

$$= \frac{E_0^2}{8\pi}\left[\cos^2 kx \cos^2 ky \cos^2 \omega t + \frac{1}{2}\sin^2 \omega t \left(\cos^2 kx \sin^2 ky + \sin^2 kx \cos^2 ky\right)\right].$$

Energy current $= \vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B}) = \frac{cE_0^2}{4\sqrt{2}\pi}\sin\omega t\cos\omega t\left[\cos^2 kx \cos ky \sin ky\ \hat{y} + x \leftrightarrow y\right].$

Now $d/dt(\sin^2\omega t) = -\frac{d}{dt}(\cos^2\omega t) = 2\omega \sin\omega t \cos\omega t$
$= \omega \sin 2\omega t,$

so $\frac{\partial u}{\partial t} = \frac{E_0^2 \omega}{16\pi}\sin 2\omega t\left[\cos^2 kx \sin^2 ky + \sin^2 kx \cos^2 ky - 2\cos^2 kx \cos^2 ky\right].$

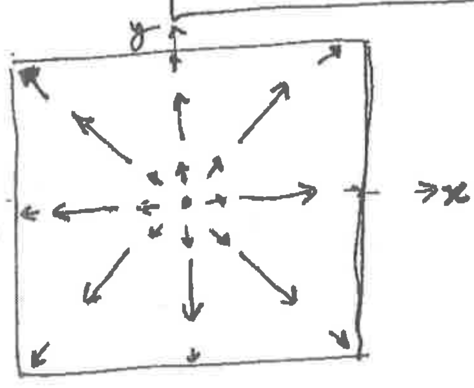
$= -\cos^2 kx \cos 2ky - \cos 2kx \cos^2 ky$.

Now, $d/dy(\sin ky \cos ky) = k(\cos^2 ky - \sin^2 ky) = k\cos 2ky$ & ditto with $y \leftrightarrow x$.

$$\therefore\ \vec{\nabla}\cdot\vec{S} = \frac{cE_0^2 \sin 2\omega t}{8\sqrt{2}\pi} k\left[\cos^2 kx \cos 2ky + \cos 2kx \cos^2 ky\right],$$

and $(\omega/16\pi) = (\sqrt{2}ck)/16\pi = ck/8\sqrt{2}\pi$, so $\boxed{\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0}$, W^5.

(b) A picture of $\vec{S}$ can be drawn in any cross-section of the box $\perp \hat{z}$. And then it looks like as drawn. A quarter cycle later it will reverse direction.



1 (15 points). A monochromatic light beam along the $+\hat{\mathbf{z}}$ direction is reflected from a large ideal mirror (100% reflectivity) in the xy plane, i.e, the $z = 0$ plane. Approximating the beam as a plane wave, the electric field in the incident wave is given by

$$\mathbf{E}_{\rm inc}(\mathbf{r}, t) = E_0 \sin(kz - \omega t + \alpha)\,\hat{\mathbf{x}}$$

where $\omega = kc$, and α is an arbitrary phase. (The light is x-polarized.)

(a) Given that at the mirror the boundary condition is $E_x = E_y = 0$, find the electric field in the reflected wave, $\mathbf{E}_{\rm ref}$, and the total field $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_{\rm inc}(\mathbf{r}, t) + \mathbf{E}_{\rm ref}(\mathbf{r}, t)$.

(b) Find the total magnetic field in the light wave.

(a). The reflected light must also be polarized along x (else $E_y \neq 0$ on surface of mirror), and must have the same freq., and travel to the left. Hence, it must have the form

$$\vec{E}_{ref}(\vec{r}, t) = E_0' \sin(kz + \omega t + \alpha')\,\hat{x},$$

Note: left moving.

where E_0' and α' are constants.

At $z = 0$, $\vec{E} = \left[-E_0 \sin(\omega t - \alpha) + E_0' \sin(\omega t + \alpha')\right]\hat{x}$.

This must vanish at all times, which implies $E_0' = E_0$, $\alpha' = -\alpha$.

$$\therefore \quad \vec{E}_{ref} = E_0 \sin(kz + \omega t - \alpha)\,\hat{x}.$$

$$\boxed{\vec{E} = \vec{E}_{inc} + \vec{E}_{ref} = 2E_0 \sin kz \cos(\omega t - \alpha)\,\hat{x}}$$

$\left(\sin(A+B) = \sin A \cos B + \cos A \sin B\right)$

(b) Faraday's law states $\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$.

$$\therefore \frac{\partial \vec{B}}{\partial t} = -c\left(\frac{\partial E_x}{\partial z}\right)\hat{y} = -2\underbrace{kc}_{=\omega} E_0 \cos kz \cos(\omega t - \alpha)\,\hat{y}.$$

It follows that $\boxed{\vec{B} = -2E_0 \cos kz \sin(\omega t - \alpha)\,\hat{y}.}$

Since $\vec{E}$ obeys the wave eqn., the Ampere-Maxwell law should be automatically obeyed if we did our calculations correctly, but we can check it explicitly also:

$$\left.\begin{aligned} \vec{\nabla} \times \vec{B} &= -\frac{\partial B_y}{\partial z}\hat{x} = -2E_0 k \sin kz \sin(\omega t - \alpha)\,\hat{x} \\ \frac{1}{c}\frac{\partial \vec{E}}{\partial t} &= -2E_0 \frac{\omega}{c} \sin kz \sin(\omega t - \alpha)\,\hat{x} \end{aligned}\right\}$$

equal, as they should be. QED