2. (15 points) For a light nucleus such as ^{40}Ca, the charge distribution is spherically symmetric, and given by

$$\rho(r) = \begin{cases} \rho_0(1 - r^2/a^2), & r \leq a, \\ 0, & r \geq a. \end{cases}$$

* Note: we have found a total charge $Q = \frac{8}{15}\pi \rho_0 a^3$. Easy to see this is correct.

Here ρ_0 and a are constants. **By direct solution of Poisson's equation,**

(a) find the potential $\phi(\mathbf{r})$ for $r \geq a$ such that $\phi \to 0$ for $r \to \infty$.

(b) Find $\phi(\mathbf{r})$ for $r < a$, making sure all physical requirements are met.

Note: For a function $f(r)$ with $\lim_{r\to 0} r f(r) = 0$, $\nabla^2 f(r) = \frac{1}{r}\frac{d^2}{dr^2}(rf)$.

(a) For $r \geq a$, $\rho = 0$, so $\frac{1}{r}\frac{d^2}{dr^2}(r\phi) = 0 \Rightarrow \frac{d^2}{dr^2}(r\phi) = 0$

$\Rightarrow r\phi = Ar + B$

$\Rightarrow \phi = A + \frac{B}{r}$. $\phi \xrightarrow{r\to\infty} 0 \Rightarrow A = 0$,

so $\boxed{\phi = B/r \quad r \geq a}$ where $B =$ const.

(b). $(r\phi)$ and $\frac{d}{dr}(r\phi)$ must be continuous at $r = a$.

$(r\phi)_a = B$; $(r\phi)'\big|_{r=a} = 0$.

Poisson's eqn. $\Rightarrow$ in Gaussian $\frac{1}{r}\frac{d^2}{dr^2}(r\phi) = -4\pi\rho_0\left(1 - \frac{r^2}{a^2}\right)$. Multiply by r & integrate once

$\frac{d}{dr}(r\phi) = -4\pi\rho_0\left(\frac{r^2}{2} - \frac{r^4}{4a^2}\right) + D \leftarrow$ const.

This must vanish at $r = a$, so $D = \pi\rho_0 a^2$.

Integrate once more:

$r\phi = -4\pi\rho_0\left(\frac{r^3}{6} - \frac{r^5}{20a^2}\right) + \pi\rho_0 a^2 r + \underbrace{F}_{\text{const.}}$.

$= 0$ ∵ there is no point charge at $r = 0$.

$\boxed{\phi(r) = \pi\rho_0 a^2 - 4\pi\rho\left(\frac{r^2}{6} - \frac{r^4}{20a^2}\right)} \leftarrow$ This is the answer

$\phi(a) = \pi\rho_0 a^2\left[1 - \frac{7}{15}\right] = \frac{8}{15}\pi\rho_0 a^2 = \frac{B}{a} \Rightarrow \rho_0 = \frac{15}{8}\frac{B}{\pi a^3}$

Write $B = Q$; $\phi(r) = \frac{15}{8}\frac{Q}{a} - \frac{Q}{a}\left(\frac{5}{4}\frac{r^2}{a^2} - \frac{3}{8}\frac{r^4}{a^4}\right)$ for $r < a$. *

Figure 1:

2.

Gauss law approach

$$Q = \int_0^a 4\pi r^2 \rho(r)\, dr$$

$$= 4\pi \int_0^a (\rho_0 r^2 - \rho_0 r^4/a^2)\, dr$$

$$= 4\pi \rho_0 \; r^3/3 - r^5/5a^2 \Big|_0^a = 4\pi \rho_0 a^3 \left(1/3 - 1/5\right)$$

$$= 8\pi \rho_0 a^3 / 15$$

$r > a$

$$\Phi_E = 4\pi r^2 E = Q/\epsilon_0 = 8\pi \rho_0 a^3 / 15\epsilon_0$$

$$E = 2 \cdot \rho_0 a^3 / 15\epsilon_0 r^2$$

$$V(r) = -\int_\infty^r E(r')\, dr'$$

$$= \frac{2 \rho_0 a^3}{15 \epsilon_0 r'} \Big|_\infty^r = \frac{2 \rho_0 a^3}{15 \epsilon_0 r}$$

Check $r = a$ $\quad V(R) = \dfrac{Q}{4\pi\epsilon_0 a} = \dfrac{2 \rho_0 a^3}{15 \epsilon_0 a}$

$r < a$ within the sphere

$$Q_{enc} = \int_0^r 4\pi r'^2 \rho(r')\, dr' = 4\pi \rho_0 \int_0^r \left(r'^2 - \frac{r'^4}{a^2}\right) dr'$$

$$= 4\pi \rho_0 \left(\frac{r^3}{3} - \frac{r^5}{5a^2}\right)$$

1.

$\downarrow \nabla^2$ if V indep of θ, ϕ

a) $r \geq a \quad \rho = 0 \qquad \frac{1}{r}\frac{d^2}{dr^2}(rV) = 0$

$$d^2/dr^2 (rV) = 0$$

$$rV = Ar + B$$

$$V = A + B/r$$

$$\boxed{V = B/r} \quad \text{if } V \to 0 \text{ at } r = \infty$$

b) can we do same for $r < a$? No!

Because $\rho \neq 0$ inside sphere.

Instead $\qquad \frac{1}{r}\, d^2/dr^2 (rV) = -\rho_0 (1 - r^2/a^2)/\epsilon_0$

$$d^2/dr^2 (rV) = -\rho_0/\epsilon_0 \left[r - r^3/a^2 \right]$$

$$\frac{d}{dr}(rV) = -\rho_0/\epsilon_0 \left[\tfrac{1}{2} r^2 - \tfrac{1}{4} r^4/a^2 \right] + C$$

$\frac{d^2}{dr^2}(rV) = \frac{\rho(r)\, r}{\epsilon_0}$

$\Updownarrow$

Analogy:

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = [E - V(x)]\psi(x) \longleftrightarrow$ rV and $d/dr(rV)$ continuous at $r = a$

$\psi, \frac{d\psi}{dx}$ continuous at bdys

From part (a) $\quad rV = B$

$$\frac{d}{dr}(rV) = 0$$

$$\therefore \quad -\rho_0/\epsilon_0 \left[\tfrac{1}{2} a^2 - \tfrac{1}{4} a^2 \right] + C = 0$$

$$C = \rho_0 a^2 / 4\epsilon_0$$

1'

We have

$$\frac{1}{dr}(rV) = \rho_0/\epsilon_0 \left\{ \frac{a^2}{4} - \frac{1}{2} r^2 + \frac{1}{4}\frac{r^4}{a^2} \right\}$$

$$rV = \rho_0/\epsilon_0 \left\{ \frac{a^2}{4} r - \frac{1}{6} r^3 + \frac{1}{20} r^5/a^2 \right\} + C'$$

must vanish because V cannot diverge at $r=0$ (no pt charge there)

$$V = \frac{\rho_0}{\epsilon_0} \left\{ \frac{a^2}{4} - \frac{1}{6} r^2 + \frac{1}{20}\frac{r^4}{a^2} \right\}$$

Can compare by $Q = \dfrac{8\pi \rho_0 a^3}{15}$

$$V = \frac{15 Q}{8\pi a^3} \frac{1}{\epsilon_0} \left\{ \frac{a^2}{4} - \frac{1}{6} r^2 + \frac{1}{20}\frac{r^4}{a^2} \right\}$$

$$= \frac{Q}{4\pi\epsilon_0} \left\{ \frac{1}{a}\frac{15}{8} - \frac{5}{4}\frac{r^2}{a^3} + \frac{3}{8}\frac{r^4}{a^5} \right\}$$

Finally can get B

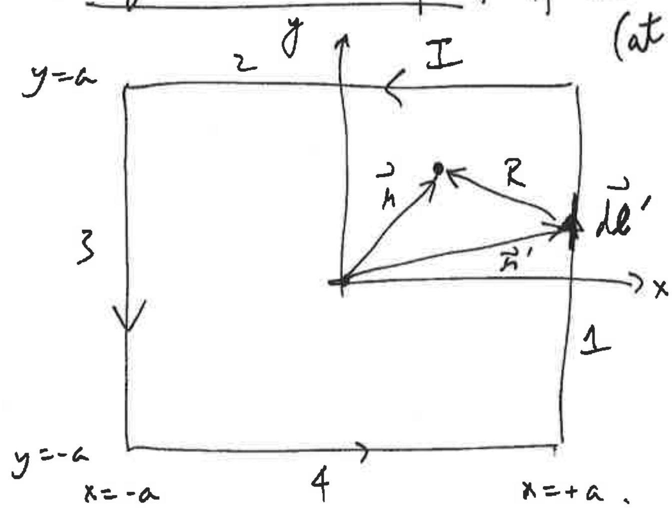
$$V(r=a) = \frac{Q}{4\pi\epsilon_0} \left\{ \frac{15}{8}\frac{1}{a} - \frac{5}{4}\frac{1}{a} + \frac{3}{8}\frac{1}{a} \right\}$$

$$\frac{1}{a}\left\{ \frac{6}{4} - \frac{5}{4} \right\}$$

$$\frac{1}{4a}$$

$$\Rightarrow B/a = \frac{Q}{4\pi\epsilon_0}\frac{1}{4a} \qquad B = \frac{Q}{16\pi\epsilon_0}$$

3. Square current loop; field in plane of loop.
(at center and everywhere else)



Use the Biot-Savart law. $\vec{dl}' \times \vec{r} \parallel \hat{z}$, so field in plane is along $\hat{z}$.

For leg 1, $(d\vec{l}' \times \vec{R}) = \hat{z}\,(a-x)\,dy'$.
So, $\vec{B}$ field due to leg 1 is

$$B_z^{(1)} = \frac{I}{c}\int_{-a}^{a} \frac{(a-x)}{\left[(a-x)^2+(y-y')^2\right]^{3/2}}\,dy'.$$

Integrate via the substitution $y'-y = |a-x|\tan\theta$.

Result is
$$B_z^{(1)} = \frac{I}{c}\,\frac{1}{(a-x)}\left\{\frac{a-y}{\sqrt{(a-y)^2+(a-x)^2}} + \frac{a+y}{\sqrt{(a+y)^2+(a-x)^2}}\right\}$$
$\leftarrow$ Call this $F(x,y)$ $\longrightarrow$

For leg 2, switch $x \,\&\, y$; leg 3 $x \to -x$; leg 4 is obtained by $y \to -y$ in leg 2 result

$$\therefore\ B_z(x,y) = \frac{I}{c}\left[F(x,y) + F(y,x) + F(-x,y) + F(-y,x)\right]$$

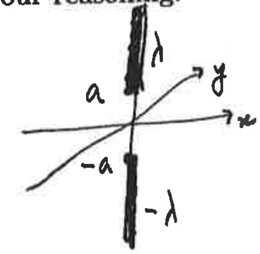
$\longrightarrow$ At center $B_z(0,0) = \left(4\sqrt{2}\, I/ac\right) \equiv B_0$ $\leftarrow$

Figure 3:

Winter 2013-14 Electricity and Magnetism (414-1) Feb. 3, 2014

Mini-Test 3

Give reasons for all answers, and show enough work that I can follow your reasoning.



1. (15 points) A wire carrying a charge per unit length $+\lambda$ lies along the z axis from $z = +a$ to ∞, and a second wire with charge per unit length $-\lambda$ runs from $z = -a$ to $-\infty$.

(a) Find the electrostatic potential ϕ on the z axis for $-a < z < a$.

(b) Expand the result in (a) in powers of z/a, going up to terms of order z^3.

(c) Use the result of (b) to find $\phi(\mathbf{r})$ near the origin, going up to terms of degree 3 in x, y, and z.

(a) $$\phi(z) = \lambda \int_a^\infty \frac{dz'}{(z'-z)} - \lambda \int_{-\infty}^{-a} \frac{dz'}{z-z'}$$ Replace $\pm\infty$ by $\pm L$ and let $L \to \infty$ at end.

$|z'-z| = z'-z \; z'>z$, $= z-z' \; z>z'$

$$= \lambda \ln(z'-z)\Big|_a^L + \lambda \ln(z-z')\Big|_{-L}^{-a}$$ (Limits are on z')

$$= \lambda \ln \frac{L-z}{a-z} \cdot \frac{z+a}{z+L} \to \lambda \ln \frac{a+z}{a-z} \quad \text{as } L \to \infty.$$

Another example of: From ϕ on axis extend to off axis

(b) Expand the logs: $$\phi(z) \simeq 2\lambda\left(\frac{z}{a} + \frac{z^3}{3a^3} + \cdots\right) \quad *$$

(c) Near origin, ϕ must (i) satisfy $\nabla^2\phi = 0$

(ii) have cylindrical symmetry.

Term $\frac{z}{a} \to \frac{z}{a} + \alpha x + \beta y \to$ cyl. symmetry $\Rightarrow \alpha = \beta = 0$.

Note $\sqrt{x^2+y^2}$ is not a polynomial.

Term $\frac{z^3}{a^3} \to \frac{1}{a^3}\left(z^3 + \alpha z(x^2+y^2)\right)$ [Has correct symmetry; is homogeneous polynomial.]

$\nabla^2 z = 0$ — OK.

$$\nabla^2\left[z^3 + \alpha z(x^2+y^2)\right] = 6z + \alpha z(2+2) = 0 \quad \text{if } \alpha = -\frac{3}{2}.$$

So $$\phi(\vec{r}) = 2\lambda\left[\frac{z}{a}\right] + \frac{2\lambda}{3a^3}\left(z^3 - \frac{3}{2}z(x^2+y^2)\right) + \cdots$$

(OVER)

$$* \ln\left(\frac{a+z}{a-z}\right) = \ln(1+z/a) - \ln(1-z/a) =$$

Figure 2:

$$= \left(z/a - \tfrac{1}{2}z^2/a^2 + \tfrac{1}{3}z^3/a^3 + \ldots\right) - \left(-z/a - \tfrac{1}{2}\frac{z^2}{a^2} - 2\,z^3/3a^3 + \ldots\right) = \frac{2z}{a} + \frac{2z^3}{3a^3}$$

$$\ln(1+x) = \int \frac{dx}{1+x} = \int 1 - x + x^2 - x^3 \ldots dx = x - x^2/2 + x^3/3$$

$$\vec{E} = -\vec{\nabla}\phi \qquad \phi = 2\lambda\, z/a + \frac{2\lambda}{a^3}\left\{ z^3 - \tfrac{3}{2} z (x^2+y^2) \right\}$$

$$E_z = -\frac{\partial}{\partial z}\phi = -\frac{2\lambda}{a} - \frac{2\lambda}{a^3}\left\{ 3z^2 - \tfrac{3}{2}(x^2+y^2) \right\}$$

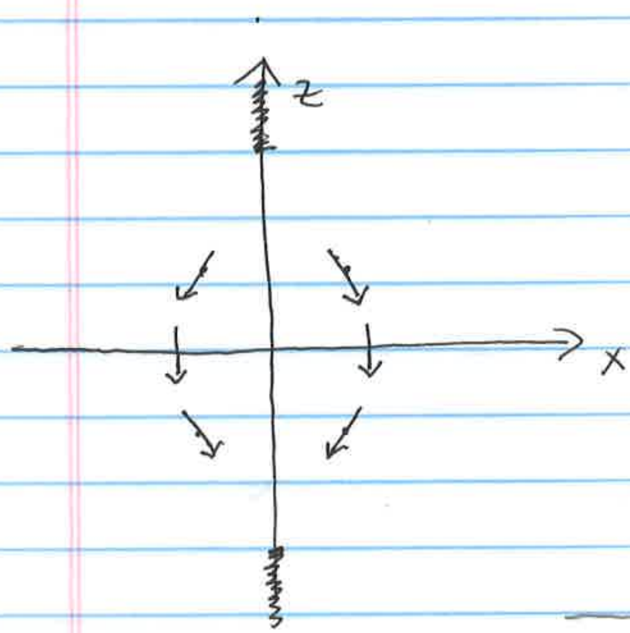
$$= -\frac{2\lambda}{a} - \frac{3\lambda}{a^3}\left\{ 2z^2 - x^2 - y^2 \right\}$$

$$E_x = -\frac{\partial}{\partial x}\phi = -\frac{2\lambda}{a^3}\left\{ -3xz \right\} = +\frac{6\lambda}{a^3} xz$$

$$E_y = \qquad = +\frac{6\lambda}{a^3} yz$$

<u>check</u> $\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$

$$= \frac{6\lambda}{a^3} z + \frac{6\lambda}{a^3} z + \left(\frac{-12\lambda}{a^3} z \right) = 0$$



in xz plane $y=0$ $E_y = 0$

$$E_z \sim -\frac{2\lambda}{a}$$

$$E_x = \frac{6\lambda}{a^3} xz \quad \begin{matrix} >0 & \text{quads } 1,3 \\ <0 & \text{quads } 2,4 \end{matrix}$$

✓✓

$$\nabla^2 (x^2+y^2)^{1/2}$$

$$= \frac{d}{dx}\left[\frac{1}{2} 2x (x^2+y^2)^{-1/2} \right] + (x \to y)$$

$$= (x^2+y^2)^{-1/2} - x^2 (x^2+y^2)^{-3/2} + (x \to y)$$

$$= (x^2+y^2)^{-3/2}\left[x^2+y^2 - x^2 + x^2+y^2 - y^2 \right] \neq 0$$

Thus $E = \frac{1}{4\pi r^2} \frac{Q_{enc}}{\epsilon_0} = \frac{\rho_0}{\epsilon_0}\left(\frac{r}{3} - \frac{r^3}{5a^2}\right)$

additional work done from $r=a$ to $r<a$ is

$$\Delta V = -\int_a^r E(r')\,dr'$$

$$= -\int_a^r \frac{\rho_0}{\epsilon_0}\left(\frac{r'}{3} - \frac{r'^3}{5a^2}\right) dr'$$

$$= -\rho_0/\epsilon_0 \left\{ \frac{r'^2}{6} - \frac{r'^4}{20a^2} \right\}_a^r$$

$$= \rho_0/\epsilon_0 \left\{ \left(\frac{a^2}{6} - \frac{a^2}{20}\right) - \left(\frac{r^2}{6} - \frac{r^4}{20a^2}\right) \right\}$$

Including V at a and converting to $Q = 8\pi\rho_0 a^3/15$ gives

$$V = \frac{Q}{4\pi\epsilon_0 a} + \frac{1}{\epsilon_0}\left\{ \frac{15Q}{8\pi}\frac{1}{a^3} a^2 \underbrace{\left(\frac{1}{6} - \frac{1}{20}\right)}_{\frac{30-3}{60}\ \ 7/60} - \frac{15Q}{8\pi a^3}\left(\frac{r^2}{6} - \frac{r^4}{20a^2}\right) \right\}$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{a}\left\{1 + \frac{7}{8}\right\} - \frac{Q}{4\pi\epsilon_0}\left(\frac{5}{4}\frac{r^2}{a^3} - \frac{3}{8}\frac{r^4}{a^5}\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a}\frac{15}{8} - \left(\frac{5}{4}\frac{r^2}{a^3} - \frac{3}{8}\frac{r^4}{a^5}\right)\right\}$$