FINAL EXAM
PHYSICS 200B, WINTER 2017
ELECTRICITY AND MAGNETISM

Intstructions: Do only six of the nine problems below!

[1.] A wire carrying charge per unit length $+\lambda$ lies along the z axis from $z = +a$ to $z = \infty$. A second wire carrying charge per unit length $-\lambda$ lies along the z axis from $z = -a$ to $z = -\infty$.
(a) Find the electrostatic potential ϕ on the z axis for $-a < z < +a$.
(b) Expand the result in (a) in powers of z/a up to order z^3.
(c) Find the potential to third order in x, y, z near the origin. (This part of the problem requires a little cleverness. Proceed on to the rest of the exam if inspiration does not strike you.) Hints: (i) What does the symmetry of the problem tell you about how x, y should enter your answer? (ii) You already know the coefficient of the z^3 term in ϕ from part (b). The condition $\nabla^2\phi = 0$ yields information about the coefficients of other possible third degree monomials in terms of your known z^3 coefficient.

[2.] Consider the vector potential $\vec{A}(\vec{r}) = A_0\, e^{-(x^2+y^2+z^2)/a^2}\, \hat{z}$, where A_0 and a are constants.
(a) Find and sketch the corresponding magnetic field.
(b) Can this be a magneto*static* field? If yes, find the current distribution that would give rise to it, and if not, explain why not.
(c) Is the vector potential given in the Coulomb gauge? If not, transform it to the Coulomb gauge to the best of your ability.

[3.] (a) What equation defines the 'Coulomb gauge'?
(b) Derive the relation

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3\vec{r}\,' \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

from Maxwell's equation $\nabla \times \vec{B} = \mu_0 \vec{J}$, the relation between $\vec{B}$ and $\vec{A}$, and the Coulomb gauge condition. Making an analogy with the electrostatic problem of getting ϕ from ρ can help make your solution convincing.

[4.] Use Ampere's law to compute the magnetic field $\vec{B}$ of a long straight wire of radius a carrying a current I which is uniformly distributed across its cross section. Give the magnitude of $\vec{B}$ for cases $r < a$ and $r > a$, and also the direction of $\vec{B}$.

[5.] (a) Use the law of Biot and Savart to compute the magnetic field $\vec{B}$ of an infinitely thin long straight wire. (Do *not* use Ampere's Law!)
(b) Compute the magnetic field at the center of a current loop.

[6.] Compute the vector potential $\vec{A}$ of an infinitely thin long straight wire from

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3\vec{r}\,' \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}.$$

Take the curl and get the magnetic field.

[7.] A sphere of radius R has potential $V(r=R,\theta) = V_0 + V_1\cos\theta + V_2\cos^2\theta$ on its surface. Determine the potential outside the sphere. There are no charges present for $r > R$. You may find the forms of the Legendre polynomials

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3)$$

useful.

[8.] The series expression for the Bessel functions is,

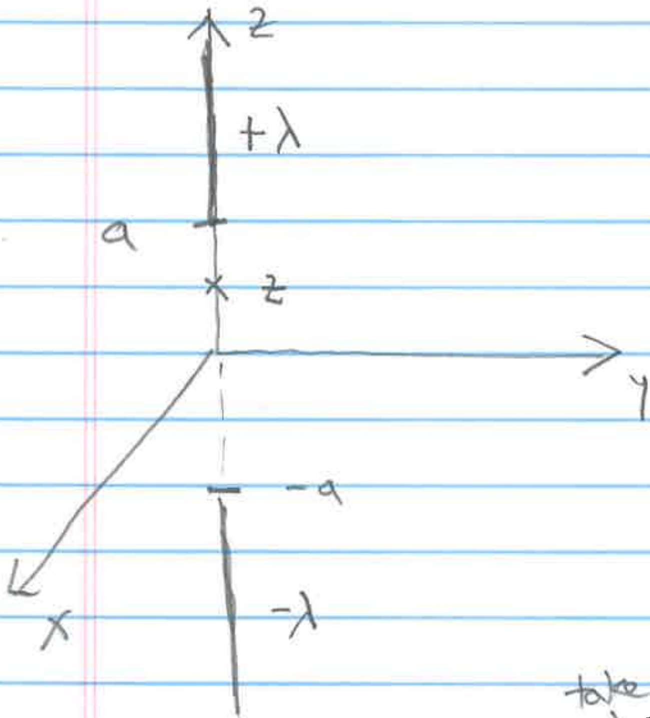
$$J_n(x) = \sum_{s=0}^{\infty} \frac{1}{s!\,(s+n)!}(-1)^s \left(\frac{x}{2}\right)^{2s+n}$$

(a) Compute $J_0(x)$ and $J_1(x)$ up to order x^6.
(b) If you just keep the expression for $J_0(x)$ up to order x^2, where does J_0 vanish? The correct value for the first root of $J_0(x)$ is 2.405.
(c) One idea for improving your value from (b) is to keep terms up to order x^4 in the expression for $J_0(x)$. If you do, where does J_0 vanish now? (Hint: Setting $u = x^2$ gives you a quadratic equation to solve.) Did your answer get much better?
(d) A second idea for improving your root is to begin with the value from (b) and use Newton's method $x_{\text{new}} = x_{\text{old}} - J_0(x_{\text{old}})/J_0'(x_{\text{old}})$. which gives a new guess x_{new} for a root from an old guess x_{old}. What do you get from this procedure, if you use the $o(x^6)$ polynomials for J_0 and its derivative? Are you happier?

[9.] Compute the magnetic field $\vec{B}$ of a magnetic dipole given $\vec{A} = (\mu_0/4\pi)\,\vec{m}\times\vec{r}/r^3$. Be complete. Provide *all* mathematical details!

P200B Final Exam Solutions

Winter 2017



$$\phi(0,0,z) = \int_a^\infty \frac{\lambda\, dz'}{\epsilon_0} \frac{1}{z'-z} + \int_{-\infty}^{-a} \frac{(-\lambda)\, dz'}{\epsilon_0} \frac{1}{z-z'}$$

because $|z'-z|$ is $z'-z$ for $z'>a$ and $z'<a$

$|z'-z| = z-z'$ for $z'<-a$ and $z'>-a$

take $L\to\infty$ at end

$$\phi(0,0,z) = \frac{\lambda}{\epsilon_0} \ln(z'-z)\Big|_a^L + \frac{\lambda}{\epsilon_0} \ln(z-z')\Big|_{-L}^{-a}$$

$$= \frac{\lambda}{\epsilon_0}\left\{ \ln\frac{L-z}{a-z} + \ln\frac{z+a}{z+L} \right\}$$

$$\to \ln\frac{(L-z)(z+a)}{(a-z)(z+L)} \to \ln\left(\frac{z+a}{a-z}\right)$$ as $L\to\infty$

$$\ln(1+x) = \int^x \frac{dx'}{1+x'} = \int^x (1-x'+x'^2-x'^3)\,dx' = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$\ln(z+a) = \ln a + \ln\left(1+\frac{z}{a}\right) = \ln a + \frac{z}{a} - \frac{z^2}{2a^2} + \frac{z^3}{3a^3}$$

$$\ln(a-z) = \ln a + \ln\left(1-\frac{z}{a}\right) = \ln a - \frac{z}{a} - \frac{z^2}{2a^2} - \frac{z^3}{3a^3}$$

Thus $\quad \phi(0,0,z) = \frac{2\lambda}{\epsilon_0}\left\{ \frac{z}{a} + \frac{z^3}{3a^3} + \dots \right\}$

Because of azimuthal symmetry $\phi(x,y,z)$ can involve x and y only in the combination x^2+y^2. If we want up to $O(3)$ in x,y,z we have

$$\phi(x,y,z) = \frac{2\lambda}{\epsilon_0}\left\{ \frac{z}{a} + \frac{z^3}{3a^3} + cz(x^2+y^2) \right\}$$

Computing $\nabla^2\phi$ gives

$$\nabla^2\phi = \frac{2\lambda}{\epsilon_0}\left\{ 0 + \frac{2z}{a^3} + cz4 \right\} = 0$$

$$\rightarrow \quad 4c = -2/a^3$$

Thus $\quad \phi(xyz) = \frac{2\lambda}{\epsilon_0}\left\{ \frac{z}{a} + \frac{z^3}{3a^3} - \frac{z}{2a^3}(x^2+y^2) \right\}$

2-1

[2.] Consider the vector potential $\mathbf{A}(\mathbf{r}) = A_0 e^{-(x^2+y^2+z^2)/a^2}\hat{\mathbf{z}}$, where A_0 and a are constants.
(a) Find and sketch the corresponding magnetic field.
(b) Can this be a magneto*static* field? If yes, find the current distribution that would give rise to it, and if not, explain why not.
(c) Is the vector potential given in the Coulomb gauge? If not, transform it to the Coulomb gauge to the best of your ability.

Solution

Taking the curl,

$$\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}} = \frac{2A_0}{a^2}\left(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}\right) e^{-r^2/a^2} = \frac{2A_0}{a^2}\left(\hat{\mathbf{z}} \times \mathbf{r}\right) e^{-r^2/a^2} \tag{4}$$

The field looks roughly like that of a toroidal solenoid. In direction, it circulates around the z axis. It has a maximum at distance $r = a/\sqrt{a}$.

Magnetostatic requires the divergence of the current density must vanish, $\nabla \cdot \vec{\mathbf{J}} = 0$, since otherwise the continuity equation $\partial\rho/\partial t = \nabla \cdot \vec{\mathbf{J}} = 0$, would require the charge density have a nonzero time derivative. We get the current density from the Maxwell equation

$$\frac{4\pi}{c}\vec{\mathbf{J}} = \vec{\nabla} \times \vec{\mathbf{B}} = \frac{4A_0}{a^4}\left(xz\hat{\mathbf{x}} + yz\hat{\mathbf{y}} + (a^2 - x^2 - y^2)\,\hat{\mathbf{z}}\right) e^{-r^2/a^2} \tag{5}$$

It is straightforward to show $\nabla \cdot \vec{\mathbf{J}} = 0$. Thus the field is magnetostatic.

To see if $\vec{\mathbf{A}}$ is in the Coulomb gauge, we compute $\nabla \cdot \vec{\mathbf{A}}$ and find

$$\nabla \cdot \vec{\mathbf{A}} = -\frac{2A_0}{a^2} z e^{-r^2/a^2} \tag{6}$$

So $\vec{\mathbf{A}}$ is *not* in the Coulomb gauge. To put it in the Coulomb gauge we need

$$0 = \nabla \cdot \vec{\mathbf{A}}' = \nabla \cdot (\vec{\mathbf{A}} + \nabla\Lambda) = \nabla \cdot \vec{\mathbf{A}} + \nabla^2\Lambda) \tag{7}$$

So that we seek Λ obeying,

$$\nabla^2\Lambda = \frac{2A_0}{a^2} z e^{-r^2/a^2} \tag{8}$$

We can solve this formally by using the Green's function for the Poisson equation,

$$\Lambda(r) = -\frac{2A_0}{4\pi a^2}\int d^3r' \, \frac{z' e^{-r'^2/a^2}}{|r - r'|} \tag{9}$$

We will not be able to do the integral completely, but we can make considerable progress. Let's begin with

$$\frac{1}{|r - r'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\hat{r})\, Y_{lm}^*(\hat{\mathbf{r}}') \tag{10}$$

2-2

We know that z' obeys Laplace's equation. In fact, it can be written as a linear combination

$$z' = r' \sum_{m=-1}^{+1} a_m Y_{1m}(\hat{\mathbf{r}}') \tag{11}$$

From the orthogonality of the Legendre polynomials the integration over d^3r' will only allow the $l = 1$ terms in the sum to survive, and their linear combination will in fact recombine to give z. In the end,

$$\Lambda(r) = -\frac{2A_0}{3a^2}\frac{z}{r}\int_0^\infty \frac{r_<}{r_>^2}\, e^{-r'^2/a^2}\, r'^3\, dr'. \tag{12}$$

Separating the integral into two spatial regions,

$$\Lambda(r) = -\frac{2A_0}{3a^2}\frac{z}{r}\left(\int_0^r \frac{1}{r^2}\, e^{-r'^2/a^2} r'^4 dr'. + \int_r^\infty r\, e^{-r'^2/a^2} r' dr'\right) \tag{13}$$

We can integrate by parts to reduce the powers of r' and get

$$\Lambda(r) = -\frac{A_0 a^2}{2}\frac{z}{r^2}\left(re^{-r'^2/a^2} - \int_0^r e^{-r'^2/a^2} dr'\right) \tag{14}$$

There is nothing we can do about this final integral except give it its usual name, the 'error function'.

3-1

a) Coulomb gauge $\Longleftrightarrow$ $\vec{\nabla}\cdot\vec{A} = 0$

b) $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$

$$\nabla\times(\vec{\nabla}\times\vec{A}) = \mu_0 \vec{J}$$

$$-\nabla^2\vec{A} + \nabla(\underbrace{\vec{\nabla}\cdot\vec{A}}_{\phi \text{ in Coulomb gauge}}) = \mu_0 \vec{J}$$

$$\nabla^2\vec{A} = -\mu_0 \vec{J}$$

looks like $\nabla^2\phi = -\rho/\epsilon_0$

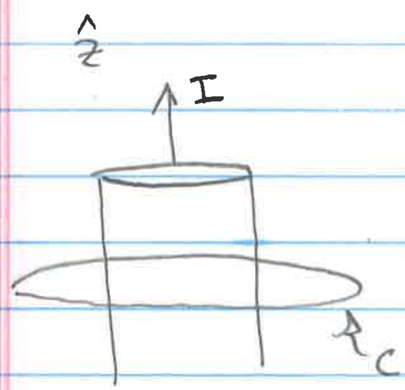
whose sol'n we know $\phi(r) = \int \frac{\rho(r')\,d^3r'}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|}$

$\therefore$ by analogy

$$\vec{A} = \int \frac{\mu_0}{4\pi} \frac{\vec{J}(\vec{r}')\,d^3r'}{|\vec{r}-\vec{r}'|}$$

4-1

Outside wire



$$\oint B \cdot d\ell = \mu_0 I_{enclosed}$$

$2\pi r|\vec{B}|$ = I (total current)

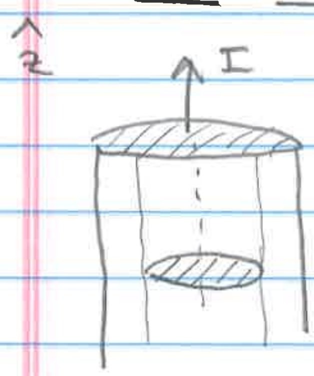
$$\Rightarrow |\vec{B}| = \mu_0 I / 2\pi r$$

direction of $\vec{B}$ is along $\hat{e}_\phi$

ie tangent to circular contour C

sense is given by right hand rule

Inside wire



$$\oint \vec{B} \cdot \vec{d\ell} = \mu_0 I_{enclosed}$$

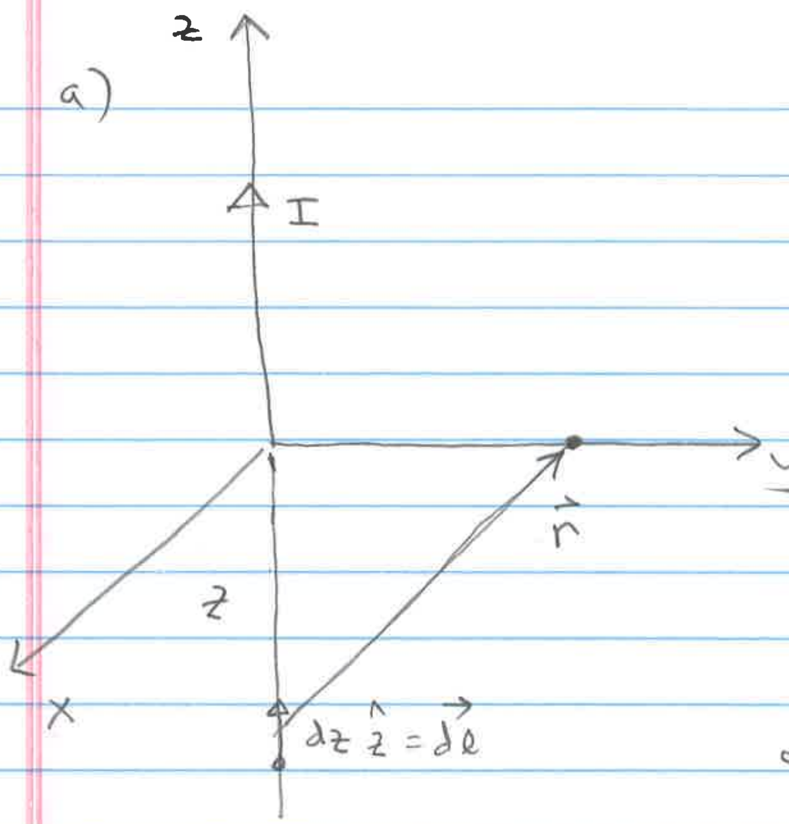
$$2\pi r |\vec{B}| = \mu_0 I \frac{\pi r^2}{\pi a^2}$$

$$|\vec{B}| = \frac{\mu_0}{2\pi} \frac{r}{a^2}$$

direction of $\vec{B}$ is same as for outside wire, ie tangent to C

5-1

a)



Evaluate at a point on the y-axis

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\vec{d\ell} \times \vec{r}}{r^3}$$

$$\vec{d\ell} = \hat{z}\,dz$$

$$\vec{r} = y\hat{y} - z\hat{z}$$

$$\vec{d\ell} \times \vec{r} = -y\hat{x}\,dz$$

$$r = (y^2+z^2)^{1/2}$$

$$\vec{B} = \int_{-\infty}^{\infty} \frac{\mu_0}{4\pi} \frac{-y\hat{x}}{(y^2+z^2)^{3/2}}\,dz$$

$$z = y\tan\theta$$

$$dz = y\sec^2\theta\,d\theta$$

$$y^2+z^2 = y^2(1+\tan^2\theta) = y^2\sec^2\theta$$

$$= -\hat{x}\frac{\mu_0}{4\pi}\int_{-\pi/2}^{\pi/2} \frac{y^2\sec^2\theta\,d\theta}{y^3\sec^3\theta\,d\theta}$$

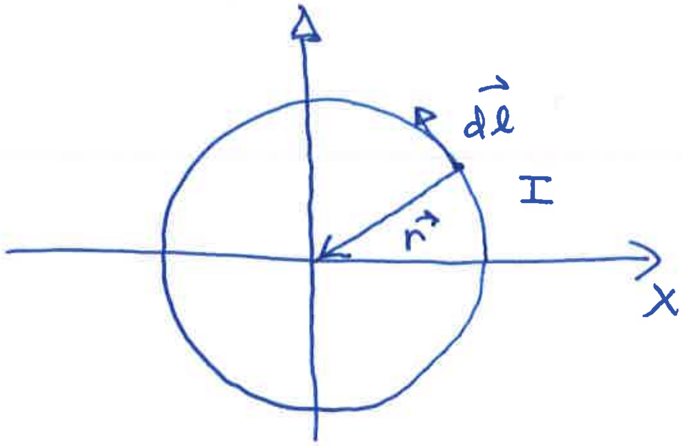
$$= -\frac{\mu_0}{4\pi y}\hat{x}\underbrace{\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta}_{2} = -\frac{\mu_0}{2\pi y}\hat{x}$$

correct direction by right hand rule

Correct also since Agrees w/ Ampere law (see problem 4)

5-2

b)



$$\vec{dB} = \frac{\mu_0 I}{4\pi} \frac{\vec{d\ell} \times \vec{r}}{r^3}$$

$$\vec{r} = (-a\cos\theta, -a\sin\theta, 0)$$

$$\vec{d\ell} = (-a\sin\theta\, d\theta, a\cos\theta\, d\theta, 0)$$

$\vec{d\ell} = -\vec{dr}$ for this geometry...

$$\vec{d\ell} \times \vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -a\sin\theta d\theta & a\cos\theta d\theta & 0 \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix} = d\theta\, \hat{z} \left(a^2\sin^2\theta + a^2\cos^2\theta\right)$$

$$= a^2 d\theta\, \hat{z}$$

actually obvious... since $\vec{d\ell} \perp \vec{r}$

$$\vec{B} = \int_0^{2\pi} \vec{dB} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a^2\, d\theta\, \hat{z}}{a^3} = \frac{\mu_0 I}{2a} \hat{z}$$

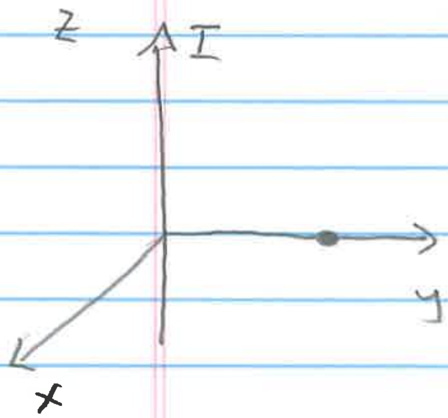
$r = a$ always

6-1

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{J}(r')}{|r-r'|}$$

Here $\vec{J}(\vec{r}') = I \, \delta(x')\delta(y') \, \hat{z}$ assuming wire along $\hat{z}$ direction. NB this eqn for $\vec{J}$ dimensionally correct: J has units of current/area and each δ function has units of $1/L$.

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} I \iiint_{-\infty}^{\infty} dx'dy'dz' \, \frac{\delta(x')\delta(y')\,\hat{z}}{(z'^2+y^2)^{1/2}}$$



Evaluating $\vec{A}$ at point $(0, y, 0)$ on Y axis

replace limits $\pm\infty$ by $\pm L$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_{-L}^{L} dz' \, \frac{1}{(z'^2+y^2)^{1/2}} \, \hat{z}$$

Let $u = z'/y$ $\quad du = dz'/y$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_{-L/y}^{L/y} y \, du \, \frac{1}{y(1+u^2)^{1/2}} \, \hat{z}$$

@ upper limit u large $\sqrt{1+u^2} \sim u$

$$= \frac{\mu_0 I}{2\pi} \, \hat{z} \, \ln \frac{L}{y}$$

6-2

$$\vec{A}(0,y,0) = \frac{\mu_0 I}{2\pi} \ln L/y \ \hat{z}$$

Take $\vec{\nabla}\times$ to get $\vec{B}$

$$\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & \frac{\mu_0 I}{2\pi} \ln L/y \end{vmatrix}$$

$\ln L - \ln y$

$$= -\hat{x}\,\frac{\mu_0 I}{2\pi y}$$

which is correct!

(Ampere's Law
see problem 4 or Biot Savart, problem 5)

This problem has azimuthal symmetry so

$$\ast \quad V(r,\theta) = \sum_{\ell=0}^{\infty} \left\{ A_\ell r^\ell + B_\ell r^{-(\ell+1)} \right\} P_\ell(\cos\theta)$$

We are interested in V outside sphere and hence

$A_\ell = 0$ if V is not to diverge as $r \to \infty$

Now notice that

$$V_0 + V_1 x + V_2 x^2 = a\,P_0(x) + b\,P_1(x) + c\,P_2(x)$$

$$= a + bx + \tfrac{3}{2}cx^2 - \tfrac{1}{2}c$$

$$\rightarrow \quad \begin{aligned} V_0 &= a - \tfrac{1}{2}c = a - \tfrac{1}{3}V_2 \\ V_1 &= b \\ V_2 &= \tfrac{3}{2}c \end{aligned}$$

So $\quad V_0 + V_1\cos\theta + V_2\cos^2\theta = \left(V_0 + \tfrac{1}{3}V_2\right) P_0(\cos\theta)$

$$+ V_1 P_1(\cos\theta) + \tfrac{2}{3}V_2 P_2(\cos\theta)$$

Thus $\quad V(r,\theta) = \left(V_0 + \tfrac{1}{3}V_2\right)\dfrac{R}{r} P_0(\cos\theta)$

$$+ \quad V_1\left(\frac{R}{r}\right)^2 P_1(\cos\theta) + \frac{2}{3}V_2\left(\frac{R}{r}\right)^3 P_2(\cos\theta)$$

has correct values at $r = R$ and obeys general form $\ast$

8-1

a) $$J_0(x) = \sum_{s=0}^{\infty} \frac{1}{s!\,s!} (-1)^s \left(\frac{x}{2}\right)^{2s}$$

$$= 1 - \frac{x^2}{4} + \frac{1}{4}\left(\frac{x}{2}\right)^4 - \frac{1}{36}\left(\frac{x}{2}\right)^6$$

$$= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{\underbrace{36\cdot 64}_{2304}}$$

$$J_1(x) = \sum_{s=0}^{\infty} \frac{1}{s!(s+1)!} (-1)^s \left(\frac{x}{2}\right)^{2s+1}$$

$$= \frac{x}{2} - \frac{1}{2}\left(\frac{x}{2}\right)^3 + \frac{1}{2\cdot 6}\left(\frac{x}{2}\right)^5$$

$$= \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{\underbrace{12\cdot 32}_{384}}$$

b) $$1 - x^2/4 = 0$$

$$\Rightarrow \quad x = 2$$

This is about 20% off the correct 2.405

c) $1 - x^2/4 + x^4/64 = 0$

$u = x^2$

$\frac{1}{64}u^2 - \frac{1}{4}u + 1 = 0$

quadratic formula

$$u = \left[1/4 \pm \sqrt{(1/4)^2 - 4(1/64)(1)} \right] 1/2(1/64)$$

$$= 32 \left\{ 1/4 \pm \sqrt{\frac{1}{16} - \frac{1}{16}} \right\}$$

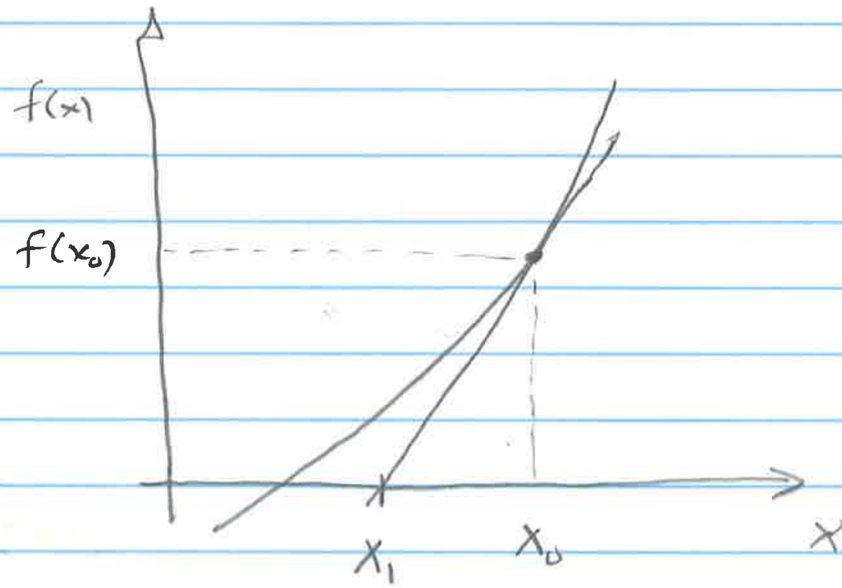
$u = 8 \rightarrow x = \sqrt{8} = 2.828$

a bit further from 2.405
than original 2.000
estimate from $J_0 = 1 - x^2/4$!

Did not help much !

8-3

d) Newton's Method



$$y - f(x_0) = f'(x_0)\left[x - x_0\right]$$

$$0 - f(x_0) = f'(x_0)\left[x_1 - x_0\right]$$

$$\therefore \quad x_1 = x_0 - f(x_0)/f'(x_0)$$

Here $\quad J_0(x) = 1 - x^2/4 + x^4/64 - x^6/\underbrace{36 \cdot 64}_{2304}$

$$J_0'(x) = -x/2 + x^3/16 - x^5/384$$

Starting at $\quad x_0 = 2 \quad$ where $\quad 1 - x^2/4 = 0$

$$x_1 = 2 - \frac{\overbrace{(1 - 1 + 1/4 - 1/36)}^{J_0(2)}}{\underbrace{(-1 + 1/2 - 1/12)}_{J_0'(2)}}$$

$$= 2 - \frac{8/36}{-7/12} = 2 + \frac{8}{21} = 2.381$$

correct answer is 2.405

This is much better!

9-1

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$$

$$B_i = (\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} A_k$$

$$= \epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\mu_0}{4\pi} \epsilon_{k\ell n} \frac{m_\ell x_n}{r^3}$$

$$1/r^3 = (x_1^2 + x_2^2 + x_3^2)^{-3/2}$$

$$\epsilon_{ijk}\epsilon_{k\ell n} = \delta_{i\ell}\delta_{jn} - \delta_{in}\delta_{j\ell}$$

$$B_i = \frac{\mu_0}{4\pi} \left\{ \frac{\partial}{\partial x_n} \frac{m_i x_n}{r^3} - \frac{\partial}{\partial x_j} \frac{m_j x_i}{r^3} \right\}$$

$$= \frac{\mu_0}{4\pi} \left\{ \frac{3m_i}{r^3} - \frac{3}{2} m_i x_n \frac{2x_n}{r^5} - \frac{m_j \delta_{ij}}{r^3} + \frac{m_j x_i}{r^5} \frac{3}{2} 2x_j \right\}$$

Because $\frac{\partial}{\partial x_n} x_n = \sum_n \frac{\partial}{\partial x_n} x_n = 3$

$x_j m_j = \vec{m} \cdot \vec{r}$

$m_j \delta_{ij} = m_i$

$x_n x_n = r^2$

$$= \frac{\mu_0}{4\pi} \left\{ \frac{3m_i}{r^3} - \frac{3m_i}{r^3} - \frac{m_i}{r^3} + \frac{3x_i \, \vec{m} \cdot \vec{r}}{r^5} \right\}$$

$$B_i = \frac{\mu_0}{4\pi} \left\{ 3 \frac{\vec{m} \cdot \vec{r}}{r^5} x_i - \frac{m_i}{r^3} \right\}$$

$$\therefore \quad \vec{B} = \frac{\mu_0}{4\pi} \left\{ \frac{3(\vec{m} \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3} \right\}$$