PHYSICS 200B, WINTER 2017
ELECTRICITY AND MAGNETISM

MIDTERM EXAM

[1.] Compute the potential $V(r,\theta,\phi)$ due to a thin ring of charge Q and radius R. Make a convenient choice of origin and orientation of your axes, and assume $r > R$. Interpret the term which falls off most slowly with $1/r$. How does your calculation change for $r < R$?

[2.] A crude model of the H_2 molecule is that the electrons form a spherical cloud of charge of radius a and the two protons are point charges inside this sphere. Find the equilibrium proton positions.

[3.] One can verify by explicit integration that the functions $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$, where $n = 1, 2, 3, \cdots$, are orthogonal and complete on the interval $x \in [0, L]$. This is the basis of Fourier expansion. Similarly, given the specific functional forms of the Legendre polynomials, they can be shown to also to be orthogonal and complete. Write a few sentences describing what more general principle might lie behind the idea of complete sets of functions. Can you make an analogy with the eigenvectors of a specific class of matrices?

[4.] Use the generating function for the Legendre polynomials

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_n t^n P_n(x)$$

to prove $P_n(1) = 1$ for all n. What can you say about $P_n(0)$?

Do only one of problems [5] or [6] below. In either case, solve the problem completely from scratch, ie starting from the appropriate partial differential equation, making a suitable guess at the form of the solution, etc.

[5.] Solve for the potential $V(x,y)$ inside a rectangular box of dimensions $0 < x < b$ and $0 < y < h$ given the boundary conditions $V(x, y = 0) = 0$; $V(x = 0, y) = 0$; $V(x = b, y) = 0$ and $V(x, y = h) = f(x)$ where $f(x)$ is an arbitrary function which vanishes at $x = 0$ and $x = b$. There is no charge inside the box. Identify the Green's function which arises in your solution.

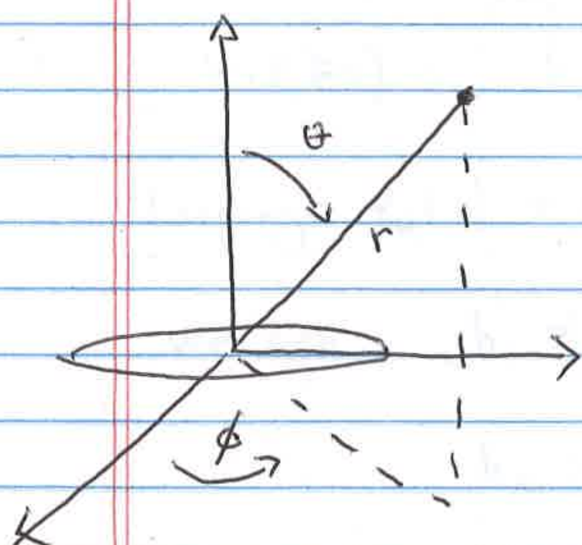
[6.] Solve for the potential $V(x,y)$ in the upper half-plane $y > 0$ if you are given the potential $V(x, y = 0) = f(x)$ along the x axis. There are no charges present. Suppose $f(x) = V_0$ is constant. What do you get for the potential $V(x,y)$? Identify the Green's function which arises in your solution.

Potentially Useful Identity:

$$(1+u)^n = 1 + nu + \frac{n(n-1)}{2}u^2 + \frac{n(n-1)(n-2)}{6}u^3 \cdots$$

P200B Winter 2017

Midterm Exam Solutions



We know

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(a_l r^l + b_l r^{-(l+1)}\right) \cdot P_l(\cos\theta)$$

for $\nabla^2 V = 0$ with azimuthal symmetry.

We can compute $V(r, \theta=0)$ (along z axis)

since all points of ring are equidistant from such points

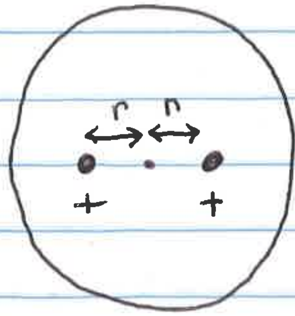
$$V(r,\theta=0) = \frac{Q}{4\pi\epsilon_0}\frac{1}{(r^2+R^2)^{1/2}}$$

assuming $r > R$

$$= \frac{Q}{4\pi\epsilon_0 r}\left(1 + R^2/r^2\right)^{-1/2}$$

$$= \frac{Q}{4\pi\epsilon_0 r}\left\{1 - \frac{1}{2}\frac{R^2}{r^2} + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{R^2}{r^2}\right)^2 + \dots\right\}$$

$$V(r,\theta=0) = \frac{Q}{4\pi\epsilon_0 r}\left\{1 - \frac{R^2}{2}\frac{1}{r^2} + \frac{3R^4}{8}\frac{1}{r^4} + \dots\right\}$$



From Gauss' law, the electric field inside a sphere of constant charge density ρ is

$$\phi_E = Q/\epsilon_0$$

$$4\pi r^2 |E| = \tfrac{1}{\epsilon_0}\left(\tfrac{4}{3}\pi r^3\right)\rho$$

In our case $\rho = Q/\tfrac{4}{3}\pi a^3$ $\qquad Q = 2e$ ← sphere has charge of 2 electrons

Putting this together,

$$|E| = \rho r/3\epsilon_0 = \frac{Q}{4\pi\epsilon_0 a^3} r = \frac{e}{2\pi\epsilon_0 a^3} r$$

The associated force $\vec{F} = \vec{E}q = -e^2/2\pi\epsilon_0 a^3 \, r \, \hat{r}$
proton

points radially inward (the − charge of e cloud pulls + proton towards origin).

The proton is also repelled by its partner, a radially outward force,

$$\vec{F}_2 = +e^2/4\pi\epsilon_0(2r)^2 \, \hat{r}$$

$$\vec{F}_{TOT} = \frac{e^2}{4\pi\epsilon_0}\hat{r}\left\{\frac{-2r}{a^3} + \frac{1}{4r^2}\right\} = 0 \quad \text{at equilibrium}$$

$$8r^3 = a^3 \qquad \boxed{r = a/2}$$

3-1

NOTE: I expect a much shorter answer from you, which, however, hits some of key points.

The collection of all (complex valued) functions $f(x)$ (infinite dimension!) can be regarded as a vector space. The inner product of two such "vectors" is $f \cdot g = \int f^*(x) g(x)\,dx$.

d^2/dx^2 is an operator (analog of a matrix) transforming one vector into another. d^2/dx^2 can be shown* to be Hermitian, ie satisfying $\int f^*(x) \frac{d^2}{dx^2} g(x)\,dx = \left(\int g^*(x) \frac{d^2}{dx^2} f(x)\,dx\right)^*$

So according to the usual rules of matrices, the eigenvectors of d^2/dx^2 are complete and orthogonal. These are $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$.

The same is true of the Legendre polynomials, which have a more complex-looking operator

$$(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + n(n+1)$$

* by integrating by parts and, to be really precise, using some information about behavior of functions on boundaries $x=0, L$ eg periodicity.

4-1

$$g(x=1,t) = 1/\sqrt{1-2t+t^2} = 1/_{1-t} = 1+t+t^2+t^3+\cdots$$

$$= \sum_n P_n(1)\,t^n$$

clearly $P_n(1) = 1$.

Similarly, setting $x=0$

$$g(x=0,t) = 1/\sqrt{1+t^2} = (1+t^2)^{-1/2}$$

$$= 1 - \tfrac{1}{2}t^2 + \tfrac{1}{2}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)t^4 + \tfrac{1}{2}\left(\tfrac{1}{3}\right)\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)t^6 + \cdots$$

$$= 1 - \tfrac{1}{2}t^2 + \tfrac{3}{8}t^4 - \tfrac{5}{16}t^6 + \ldots$$

we see that $P_n(0) = 0$ for all odd n.

This is consistent with fact that $P_n(x)$ are odd functions of x: $P_n(x) = -P_n(-x)$ for n odd.

We also see $P_0(0) = 1$ $\quad P_2(0) = -1/2$

$P_4(0) = 3/8$ $\quad P_6(0) = -5/16$

and, again, we could write a general expression if we were really motivated to do so.

5-1

$V(x,y)$ obeys the Laplace eqn

$$\nabla^2 V = \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) V = 0$$

Guess a solution $V(x,y) = r(x)\, s(y)$ ← "separation of variables"

$$s(y) \frac{d^2 r}{dx^2} + r(x) \frac{d^2 s}{dy^2} = 0$$

$$\Rightarrow \underbrace{\frac{1}{r(x)} \frac{d^2 r}{dx^2}}_{\text{function of } x \text{ only}} = \underbrace{-\frac{1}{s(y)} \frac{d^2 s}{dy^2}}_{\text{function of } y \text{ only}} = -k^2$$

function of x only ⟺ function of y only ⟹ must be constant

clearly $r(x) = \begin{matrix} \sin kx \\ \cos kx \end{matrix}$ $\quad s(y) = \begin{matrix} \sinh ky \\ \cosh ky \end{matrix}$

Since $V(x,y)$ vanishes at $x=0$ and $x=b$

we much choose $\sin kx$ for $r(x)$ and also $k = \frac{n\pi}{b}$

$$V(x,y) = \sum_n q_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi y}{b} \quad \Big\}\ \text{"superposition"}$$

where I also used fact that V vanishes at $y=0$

to eliminate $\cosh ky$ soln of $s(y)$)

We use our final bdy information

$$V(x, y=h) = f(x) = \sum_n a_n \sin\frac{n\pi x}{b} \sinh\frac{n\pi h}{b}$$

Multiply both sides by $\sin\frac{\ell\pi x}{b}$ and integrate $\int_0^b dx$

Using orthogonality. $\int_0^b \sin\frac{\ell\pi x}{b} \sin\frac{n\pi x}{b} dx = \frac{b}{2}\delta_{n\ell}$

$$\int_0^b \sin\frac{\ell\pi x}{b} f(x) dx = a_\ell \frac{b}{2} \sinh\frac{\ell\pi h}{b}$$

This determines the a_ℓ which we put back in $V(x y)$

$$V(x y) = \sum_n \frac{2}{b \sinh\frac{n\pi h}{b}} \int_0^b \sin\frac{n\pi x'}{b} f(x') dx'$$

$\sin\frac{n\pi x}{b} \sinh\frac{n\pi y}{b}$

Reorganizing

$$V(x y) = \int_0^b f(x') g(x, x', y) dx'$$

with green's function

$$g(x, x', y) \equiv \sum_n \frac{2}{b \sinh\frac{n\pi h}{b}} \sin\frac{n\pi x'}{b} \sin\frac{n\pi x}{b} \sinh\frac{n\pi y}{b}$$

We have $V(x,y)$ obeying Laplace Eqn

$$\nabla^2 V = \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) V(x,y) = 0$$

Try separation of variables $V(x,y) = r(x)s(y)$

$$s(y)\frac{d^2r}{dx^2} + r(x)\frac{d^2s}{dy^2} = 0$$

$$\underbrace{\frac{1}{r(x)}\frac{d^2r}{dx^2}}_{\text{function of } x \text{ only}} = \underbrace{-\frac{1}{s(y)}\frac{d^2s}{dy^2}}_{\text{function of } y \text{ only}} = -k^2$$

function of x only $\longleftrightarrow$ function of y only $\Rightarrow$ must be constant

We have $r(x) = e^{ikx}$

$s(y) = e^{ky}$ ← eliminate in upper half plane $y>0$ to avoid V diverging

e^{-ky}

see p 6-2 for impt "detail"

$$V(x,y) = \int_{-\infty}^{\infty} a(k) e^{ikx} e^{-ky}\, dk \qquad \text{"superposition"}$$

We are given

$$V(x,y=0) = f(x) = \int_{-\infty}^{\infty} a(k) e^{ikx}\, dk$$

Invert this Fourier integral ... $\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx = a(k)$

Putting together

$$V(x,y) = \int_{-\infty}^{\infty} dk \, \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x') e^{-ikx'} dx' \, e^{ikx} e^{-ky}$$

$$= \int dx' \, f(x') \, g(x,x',y)$$

with $g(x,x',y) \equiv \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} e^{-ky}$

In this case we can explicity compute $g(x x' y)$

Note that in selecting e^{-ky} over e^{ky} we were assuming $k>0$. we really meant $e^{-|k|y}$

$$g(x,x',y) = \int_{-\infty}^{0} \frac{dk}{2\pi} e^{ik(x-x')} e^{ky} + \int_{0}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} e^{-ky}$$

$$= \frac{1}{2\pi} \left\{ \left. \frac{e^{ky+ik(x-x')}}{y+i(x-x')} \right|_{-\infty}^{0} + \left. \frac{e^{-ky+ik(x-x')}}{-y+i(x-x')} \right|_{0}^{\infty} \right\}$$

$$= \frac{1}{2\pi} \left\{ \frac{1}{y+i(x-x')} - \frac{1}{-y+i(x-x')} \right\}$$

$$= \frac{1}{2\pi} \left\{ \frac{y-i(x-x')+y+i(x-x')}{y^2+(x-x')^2} \right\}$$

$$g(x,x',y) = \frac{y/\pi}{y^2+(x-x')^2}$$

6-3

If $f(x) = V_0$ is constant

$$V(x,y) = \int_{-\infty}^{\infty} dx' \, V_0 \, \frac{y/\pi}{y^2 + (x-x')^2}$$

change variables $u = -x + x'$

$du = +dx'$

$$V(x,y) = \int_{-\infty}^{\infty} V_0 \, {y/\pi}/{(y^2+u^2)} \, du$$

and trig substitution $u = y \tan\theta$

$du = y \sec^2\theta \, d\theta$

$y^2 + u^2 = y^2(1 + \tan^2\theta) = y^2 \sec^2\theta$

$$V(x,y) = \int_{-\pi/2}^{\pi/2} V_0 \, \frac{y/\pi \, y \sec^2\theta \, d\theta}{y^2 \sec^2\theta}$$

$$= \frac{V_0}{\pi} \int_{-\pi/2}^{\pi/2} d\theta = V_0$$

The potential is constant in the entire

upper half plane!