Magnetostatics

Electrostatics begins with Coulomb's law, the $\vec{E}$ field produced by point charge Q_1

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0} \frac{Q_1 \hat{r}}{r^2}$$

and the force it produces on a second point charge

$$\vec{F}_{1\,on\,2} = Q_2 \vec{E}_1 = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r_{12}^2} \hat{r}_{12}$$

One approach to electrostatics of more complicated charge distributions is division into small "point" charges and integration. Often this process is aided (simplified) by considering the electrostatic potential

$$V_1 = \frac{1}{4\pi\epsilon_0} Q_1/r$$

$$\vec{E}_1 = -\nabla V_1$$

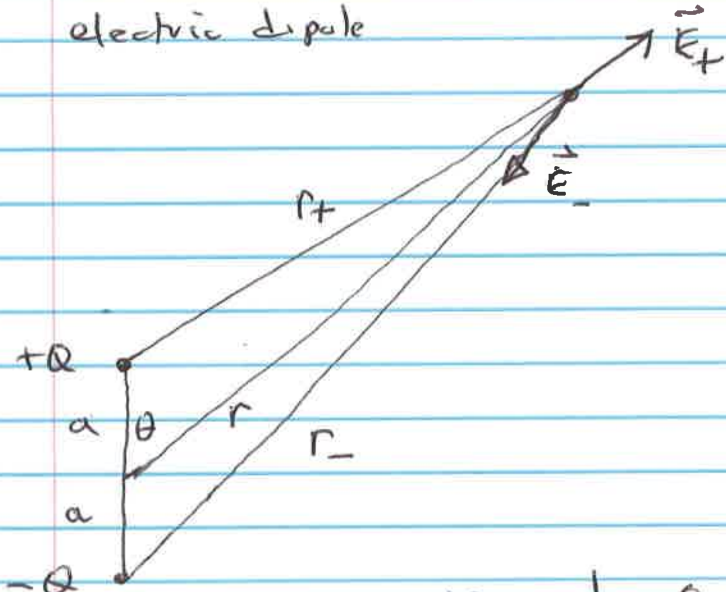
because it is easier to add scalars (the potential) than vectors.

We developed additional tricks for electrostatics by combining $\vec{\nabla}\cdot\vec{E} = 0$ with $\vec{E} = -\nabla V \Rightarrow \nabla^2 V = 0$ SOLVE LAPLACE EQN!

MS-2

Simplest Example of combining charges is

electric dipole



Challenging to add $\vec{E}_+$ and $\vec{E}_-$

However, potential is

much easier, We did

this before:

$$V = \frac{1}{4\pi\epsilon_0} Q \left\{ \frac{1}{r_+} - \frac{1}{r_-} \right\}$$

$$r_+^2 = r^2 + a^2 - 2ar\cos\theta$$

$$r_-^2 = r^2 + a^2 - 2ar\cos(\pi - \theta) = r^2 + a^2 + 2ar\cos\theta$$

$r >> a$ $\quad r_+ = r\left(1 - \frac{2a}{r}\cos\theta + a^2/r^2\right)^{1/2}$

$$\approx r\left(1 - a/r \cos\theta\right)$$

$$V = \frac{Q}{4\pi\epsilon_0} \frac{1}{r} \left\{ \frac{1}{1 - a/r\cos\theta} - \frac{1}{1 + a/r\cos\theta} \right\}$$

$$\downarrow \qquad \downarrow$$

$$1 + a/r\cos\theta \qquad 1 - a/r\cos\theta$$

$$V \approx \frac{Q}{4\pi\epsilon_0} \frac{1}{r} \frac{2a}{r}\cos\theta = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}$$

with $|\vec{p}| \equiv 2aQ$ and $\vec{p}$ pointing from $-Q$ to $+Q$

MS-3

the electric field is then

$$\vec{E} = -\vec{\nabla}\,\frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}$$

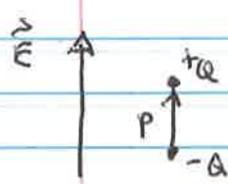
$$E_x = -\frac{\partial}{\partial x}\,\frac{p_x x + p_y y + p_z z}{4\pi\epsilon_0 (x^2+y^2+z^2)^{3/2}}$$

$$= -\frac{p_x}{4\pi\epsilon_0 (x^2+y^2+z^2)^{3/2}} + \frac{3}{2}\,\frac{p_x x + p_y y + p_z z}{(x^2+y^2+z^2)^{5/2}}\,2x$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0 r^3}\left\{\frac{3(\vec{p}\cdot\vec{r})\vec{r}}{r^2} - \vec{p}\right\}$$

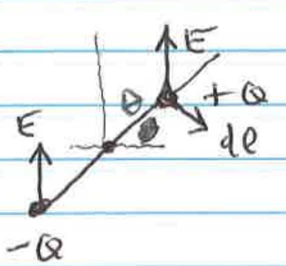
Can ask for the interaction energy of a second dipole in the field due to the first dipole, the analog of the interaction energy of 2 point charges $\frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r_{12}}$

It is $\quad -\vec{p}_2\cdot\vec{E}_1 = \frac{1}{4\pi\epsilon_0 r_{12}^3}\left\{\frac{3(\vec{p}_1\cdot\vec{r}_{12})(\vec{p}_2\cdot\vec{r}_{12})}{r_{12}^2} - \vec{p}_1\cdot\vec{p}_2\right\}$



lowest energy (obviously)

Work to rotate



Work done by us

$$dW = -2QE\,a\,d\theta\,\cos\left(\tfrac{\pi}{2}+\theta\right)$$
$$= +2QE\,a\sin\theta\,d\theta$$

$$U = \int dW = -2QEa\cos\theta$$

$$= -\vec{p}\cdot\vec{E} \qquad \vec{p} = 2Qa$$

MS-4

There are no magnetic monopoles, however elementary particles like the electron do possess a magnetic dipole. One way to begin magnetostatics is to ascribe a magnetic field due to a magnetic dipole

$$\vec{B} = \frac{\mu_0}{4\pi r^3}\left\{ \frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^2} - \vec{m} \right\}$$

and likewise an interaction energy of a dipole in a field

$$U = -\vec{m}\cdot\vec{B}$$

One can take these as "laws" analogous to Coulomb's law. We did not need a "law" for electric dipoles because it followed from Coulomb's law for electric monopoles.

MS-5

Another (more traditional) way to define magnetic fields and derive interaction energies is to postulate that moving charges q_2 experience a force when in a magnetic field $\vec{B}$

$$\vec{F} = q_2 \vec{v}_2 \times \vec{B}$$

and that a moving charge q_1 produces a magnetic field (Law of Biot and Savart)

$$\vec{B} = \frac{\mu_0}{4\pi} q_1 \frac{\vec{v}_1 \times \vec{r}}{r^3} = \frac{\mu_0}{4\pi} q_1 \frac{\vec{v}_1 \times \hat{r}}{r^2}$$

One can show that if one defines a magnetic dipole as an infinitesimal current loop  I $\quad \vec{m} = I A$

$\uparrow$ area of loop

That the Lorentz force law and the law of Biot + Savart reproduce the dipole field law and dipole-dipole interaction energy. This is a bit of cumbersome algebra because the current loop has 4 sides that need to be added correctly!

MS-6

Just as we derived Maxwell's Eqn $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 = \frac{Q\delta(r)}{\epsilon_0}$ from the field due to a point charge $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^3}\vec{r}$ (~~as~~ zero as long as $\vec{r} \neq 0$!) we can also show $\vec{\nabla}\cdot\vec{B} = 0$ from the Biot-Savart law. (It is obvious that $\vec{\nabla}\cdot\vec{B} = 0$ for the magnetic dipole field $\vec{B}$ since this is the exact same problem as for an electric dipole!)

$$\vec{B} = Q\frac{\vec{V}\times\vec{r}}{r^3}\frac{\mu_0}{4\pi}$$

$$\vec{\nabla}\cdot\vec{B} = \frac{\partial}{\partial x_i}B_i = \frac{\partial}{\partial x_i}Q\frac{(\vec{V}\times\vec{r})_i}{r^3}\frac{\mu_0}{4\pi}$$

$$= \frac{\mu_0 Q}{4\pi}\frac{\partial}{\partial x_i}\frac{\epsilon_{ijk}V_j X_k}{r^3} \qquad 1/r^3 = 1/(x_1^2+x_2^2+x_3^2)^{3/2}$$

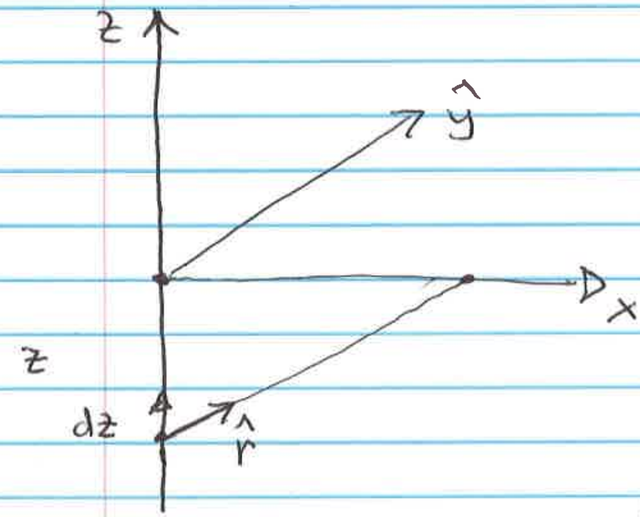
$$= \frac{\mu_0 Q}{4\pi}\left\{\frac{\epsilon_{ijk}V_j\delta_{ik}}{r^3} - \epsilon_{ijk}V_jX_k\frac{3}{2}\frac{1}{r^5}2x_i\right\}$$

vanishes because $\epsilon_{ijk} = 0$ if 2 indices identical (δ_{ik})

$$\underbrace{\epsilon_{ijk}}_{asym}\underbrace{x_k x_i}_{sym} = 0$$

MS-7

$\vec{B}$ field due to a long straight wire



$$\frac{\mu_0}{4\pi} \frac{Q\vec{v}\times\hat{r}}{r^2}$$

$$I = \frac{Q}{dt} \qquad Q\vec{v} = I \underbrace{\vec{v}dt}_{d\vec{l}}$$

$$\frac{\mu_0}{4\pi} I \frac{\vec{dl}\times\hat{r}}{r^2}$$

$$\vec{dl} = (0, 0, dz)$$

$$\vec{r} = (x, 0, -z)$$

$$\vec{dl}\times\vec{r} = (0, x\,dz, 0)$$

$$dB = \frac{\mu_0 I}{4\pi} \frac{x\,dz}{r^3} \hat{y}$$

$$B = \int_{-\infty}^{\infty} \frac{\mu_0 I x}{4\pi} \frac{dz}{(x^2+z^2)^{3/2}} \hat{y}$$

$$z = x\tan\theta$$
$$dz = x\sec^2\theta\, d\theta$$
$$x^2+z^2 = x^2(1+\tan^2\theta) = x^2\sec^2\theta$$

$$B = \frac{\mu_0 I x}{4\pi}\hat{y} \int_{-\pi/2}^{\pi/2} \frac{1}{x^2}\cos\theta\, d\theta \qquad \sin\theta\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{\mu_0 I}{4\pi x}\hat{y}\, 2$$

$$= \frac{\mu_0 I}{2\pi x}\hat{y} \qquad |\vec{B}| = \frac{\mu_0 I}{2\pi r}$$

direction "right hand rule"

This is consistent with Ampere's law, one of Maxwell's Eqns in the static case

$$\int \vec{B}\cdot d\vec{\ell} = \mu_0 I$$ ← this is the more elegant way of doing magneto-statics

$$B\,2\pi r = \mu_0 I$$

① Coulomb Law vs ② $E = -\nabla V$

$\nabla \cdot E = \rho/\epsilon_0$

$\nabla^2 V = -\rho/\epsilon_0$

$\int \qquad \phi_E = Q/\epsilon_0$

① Biot-Savart Law

② ~~[illegible]~~ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

analog of V ?!

$$\int \vec{B}\cdot d\vec{\ell} = \int \vec{\nabla}\times\vec{B}\cdot\hat{n}\,dA = \mu_0 I$$

$$I = \int \vec{j}\cdot\hat{n}\,dA$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

Recall Continuity Eqn $\quad \vec{\nabla}\cdot\vec{j} + \partial\rho/\partial t = 0$

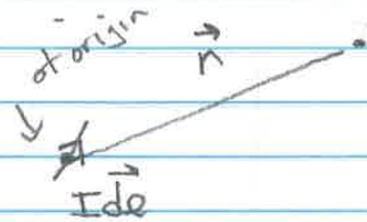
proof: $$-\frac{d}{dt}\int \rho\, dV = \oint \vec{j}\cdot\hat{n}\,dA$$

$$= \int \vec{\nabla}\cdot\vec{j}\,dV$$

MS-9

Vector potential

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \vec{\nabla}\times\vec{A}$$



$$B(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{I\vec{dl}\times\vec{r}}{r^3}$$

$I = J da$

$I\vec{dl} \Rightarrow \vec{J} da dl$

$\vec{J}(r')d^3r'$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(r')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} d^3r'$$

But $$-\nabla\frac{1}{|\vec{r}-\vec{r}'|} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

so $$\vec{B}(\vec{r}) = -\frac{\mu_0}{4\pi}\int \vec{J}(r')\times\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|} d^3r'$$

$$= \frac{\mu_0}{4\pi}\vec{\nabla}\times\int\frac{\vec{J}(r')}{|\vec{r}-\vec{r}'|}d^3r'$$

This last step follows because

$$\left(\vec{J}\times\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}\right)_i = \epsilon_{ijk}J_j\frac{\partial}{\partial x_k}\frac{1}{|r-r'|}$$

$$\left(\nabla\times\frac{J(r')}{|\vec{r}-\vec{r}'|}\right)_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{J_k(r')}{|r-r'|}$$

$$= \epsilon_{ikj}\frac{\partial}{\partial x_k}\frac{J_j(r')}{|r-r'|}$$

$$= -\epsilon_{ijk}J_j(r')\frac{\partial}{\partial x_k}\frac{1}{|r-r'|}$$

MS-9A

Review

Any vector field can be decomposed $\vec{A} = \vec{B} + \vec{C}$

$\nabla \times \vec{C} = 0$ irrotational

$\nabla \cdot \vec{B} = 0$ solenoidal

Fourier Decompose

$$\vec{A}(\vec{r}) = \int d^3k \, e^{i\vec{k}\cdot\vec{r}} \, \vec{A}(\vec{k})$$

$$= \int d^3k \, e^{i\vec{k}\cdot\vec{r}} \left\{ \underbrace{\vec{k}\cdot\vec{A}(k)\frac{\vec{k}}{k^2}}_{``\vec{A}_L(\vec{k})"} + \underbrace{\left(\vec{A}(k) - \vec{k}\cdot\vec{A}(\vec{k})\frac{\vec{k}}{k^2}\right)}_{``\vec{A}_T(\vec{k})"} \right\}$$

$$\vec{C}(\vec{r}) \equiv \int d^3k \, \vec{A}_L(k) \, e^{i\vec{k}\cdot\vec{r}}$$

$$\vec{B}(\vec{r}) \equiv \int d^3k \, \vec{A}_T(k) \, e^{i\vec{k}\cdot\vec{r}}$$

$\vec{A}(\vec{r}) = \vec{B}(\vec{r}) + \vec{C}(r)$ and

$\vec{\nabla}\cdot\vec{B} = 0$ since $\vec{k}\cdot\vec{A}_T(\vec{k}) = 0$

$\vec{\nabla}\times\vec{C} = 0$ since $\vec{k}\times\vec{A}_L(k) = 0$

MS-10

$\frac{4\pi}{c} \to \mu_0$

SI

We conclude

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(r')}{|\vec{r}-\vec{r}'|} d^3r'$$

Compare this to

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r')}{|\bar{r}-\bar{r}'|} d^3r'$$

Redo infinite straight wire using $\vec{A}$

$$\vec{J} = I\,\delta(x)\delta(y)\,\hat{z}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \frac{I\,\delta(x')\delta(y')\,\hat{z}}{(z'^2+x^2)^{1/2}} dx'dy'dz'$$

$$A_z = \frac{\mu_0 I}{2\pi} \int_0^{\infty} \frac{dz'}{(z'^2+x^2)^{1/2}}$$

$$= \lim_{L\to\infty} \frac{\mu_0 I}{2\pi} \int_0^L \frac{dz'}{[(z')^2+x^2]^{1/2}}$$

$z' = x \tanh\theta$

$dz' = x\,\mathrm{sech}^2\theta\, d\theta$

$$= \lim_{L\to\infty} \frac{\mu_0 I}{2\pi} \ln\left(\frac{2L}{x}\right)$$

See MS-10'

$$\vec{B} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & \frac{\mu_0 I}{2\pi}\ln\frac{2L}{x} \end{vmatrix}$$

$\ln\frac{2L}{x} = \ln 2L - \ln x$

$$= \frac{\mu_0 I}{2\pi x}\hat{y} \quad \checkmark\checkmark$$

MI-10A

$$\int_0^L \frac{dx}{\sqrt{x^2+c^2}} = \;? \quad \text{for } L \text{ large}$$

$$u = x/c \qquad du = dx/c$$

$$\int_0^{L/c} \frac{c\,du}{\sqrt{c^2(u^2+1)}} = \int_0^{L/c} \frac{du}{\sqrt{u^2+1}}$$

@ upper limit $\quad u^2+1 \approx u^2$

$$\int_0^{L/c} \frac{du}{u} = \ln u \Big|^{L/c} \sim \ln L/c$$

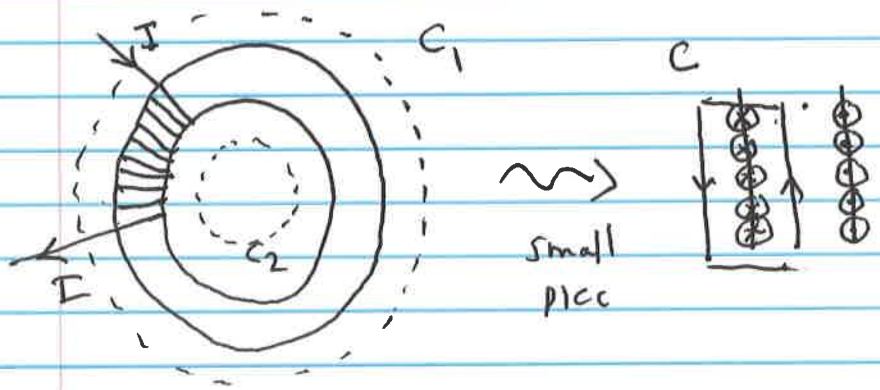
via
↙ Ampere Law

Solenoid : wind wires tightly around cylinder.

#1

Different geometries possible but consider cylinder itself

bent into torus.



By symmetry $\vec{B}$ must be cylindrically symmetric ie $\vec{B}$ same all along C_1

$$\oint_{C_1} \vec{B}\cdot d\vec{\ell} = \mu_0 I_{tot} = 0 \quad \Rightarrow \quad \vec{B} = 0$$

outside solenoid

same along C_2

Inside solenoid B must point along ~~side~~ axis if it is very long so basically a straight cylinder.

$$\oint_{c} B \cdot d\ell = BL = \mu_0 I_{tot} = \mu_0 n I L \qquad B = \mu_0 n I$$

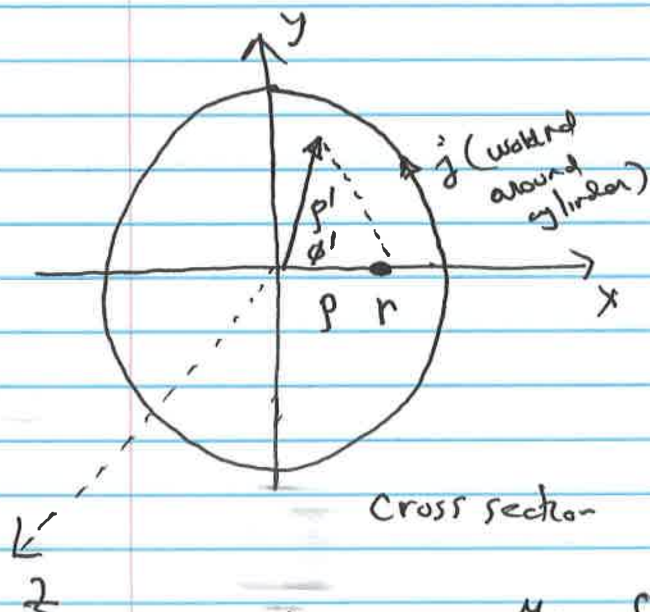
n = turns/length

S-2

Solenoid Method #2 Direct integration

of $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{j}(\vec{r}\,')}{|r-r'|} d^3r'$

z, ρ, ϕ



Cross section

Indep of z, ϕ by symmetry

so place $\vec{r}$ in xy plane

along $\hat{x}$ axis $(z=0, \phi=0)$

$$\vec{j}(r') = nI\,\delta(\rho'-a)\,\hat{e}_\phi$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_0^{2\pi} d\phi' \int \underset{a}{\rho' d\rho'} \int dz' \frac{nI\,\delta(\rho'-a)\,\hat{e}_\phi}{|r-r'|}$$

$$|\bar{r}-\bar{r}'|^2 = (z'-\underset{z}{0})^2 + \underset{a}{\rho'^2} + \rho^2 - 2\rho\underset{a}{\rho'}\cos\phi'$$

$$A_\phi(\vec{r}) = \frac{\mu_0}{4\pi} nIa \int_0^{2\pi} d\phi' \int dz' \frac{\cos\phi'}{[z'^2+\rho^2+a^2-2a\rho\cos\phi']^{1/2}}$$

$\rho, z=0, \phi=0$

using MS-10A $\quad \ln L - \ln(\rho^2+a^2-2a\rho\cos\phi')$

This part vanishes upon $\int_0^{2\pi} d\phi' \cos\phi'$

5-3

$u = \sin\phi'$

$du = \cos\phi' d\phi' \quad v = \ln(\rho^2 + a^2 - 2a\rho\cos\phi')$

$$A_\phi = -\frac{\mu_0}{4\pi} n I a \int_0^{2\pi} \cos\phi' \ln(\rho^2 + a^2 - 2a\rho\cos\phi')\, d\phi'$$

integrate by parts ...

$$= \frac{\mu_0}{4\pi} n I a \int_0^{2\pi} \sin\phi' \frac{2a\rho\sin\phi'}{\rho^2 + a^2 - 2a\rho\cos\phi'}\, d\phi'$$

$$= \frac{\mu_0}{2\pi} n I a^2 \rho \int_0^{2\pi} \frac{\sin^2\phi'\, d\phi'}{\rho^2 + a^2 - 2a\rho\cos\phi'}$$

Contour integral? circle in complex plane $z = e^{i\phi}$

$\cos\phi' = (z + 1/z)\, 1/2$

$\sin\phi' = (z - 1/z)\, 1/2i$

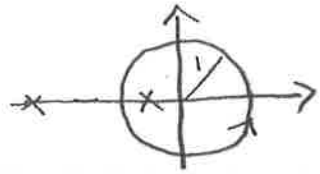
poles of denominator inside circle ... etc.

or $u = \tan(\phi'/2) \qquad du = \sec^2 \phi'/2 \; 1/2\, d\phi'$

$$A_\phi = \frac{\mu_0}{2} n I \begin{cases} \rho & \rho < a \\ a^2/\rho & \rho > a \end{cases}$$

$$\vec{\nabla} \times \vec{A} = \begin{cases} \mu_0 I n & \rho < a \\ 0 & \rho > a \end{cases}$$

RT 5



Some more examples:

RT5A first

$$\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} \qquad z = e^{i\theta} \qquad \cos\theta = \frac{1}{2}(z + 1/z)$$

$$dz = ie^{i\theta}\, d\theta$$

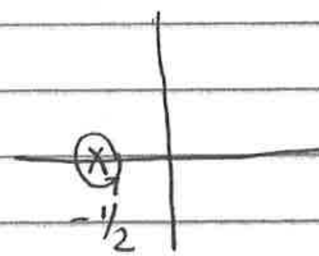
$$= \oint_C \frac{dz/iz}{5+4\left(\frac{1}{2}\right)(z+1/z)} = \frac{1}{i}\oint \frac{dz}{5z+2z^2+2}$$

$$= \frac{1}{i}\oint \frac{dz}{(2z+1)(z+2)}$$

poles at $z=-2$

$z = -1/2$

Distort contour



"residue"

evaluate $B_1 = (z-z_0)\, f(z)\Big|_{z=z_0}$

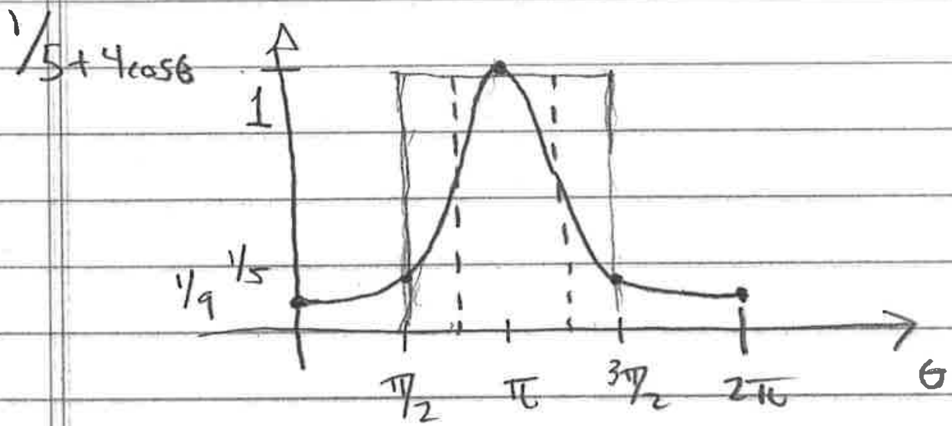
$$(z+1/2)\frac{1}{(2z+1)(z+2)}\Bigg|_{z=-1/2} = \frac{1}{2(z+2)}\Bigg|_{z=-\frac{1}{2}} = \frac{1}{3}$$

Integral is $\frac{1}{i}\, 2\pi i\left(1/3\right) = 2\pi/3$

This one you probably did not learn how to do in Math 21!

RT5A

Draw a picture of integrand



"guesstimate" $I < \pi \cdot 1 = \pi$

$I > \pi/2 \cdot 1 = \pi/2$

Gauge invariance

Electrostatic potential $V(\vec{r}) \to V(\vec{r}) + \text{const}$

doesn't alter $\vec{E} = -\nabla V$

But this is relatively trivial because no $\vec{r}$ dependence

enters the family of equivalent potentials. In contrast

for the vector potential $A' = A + \nabla\psi$ ← "gauge degree of freedom"

$$B' = \vec{\nabla} \times A' = \vec{\nabla} \times \vec{A} = B$$

but $\nabla\psi$ can depend on $\vec{r}$.

vector field

We know $\nabla \times \vec{V}$ & $\vec{\nabla} \cdot \vec{V}$ determine $\vec{V}$ (we discussed

explicitly how to break any vector field $\vec{V}$ into pieces with

$\vec{\nabla} \cdot \vec{V}_1 = 0 \quad \vec{\nabla} \times \vec{V}_1 = 0 \quad \vec{V} = \vec{V}_1 + \vec{V}_2$

In this case we know $\vec{\nabla} \times \vec{A}$ because we are attempting

to describe a particular $\vec{B}$.

$$\vec{\nabla} \cdot \vec{A}' = \vec{\nabla} \cdot \vec{A} + \nabla^2 \psi$$

one choice of gauge is to choose the $\vec{A}$ which has $\vec{\nabla} \cdot \vec{A} = 0$

"fixing the gauge"

Consider implications of Coulomb gauge with Ampere's law

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2 A + \nabla(\underbrace{\nabla\cdot A}_{\not 0})$$

$$\Rightarrow \quad \nabla^2\vec{A} = -\mu_0\vec{J} \longrightarrow \begin{cases} \nabla^2 A_x = -\mu_0 j_x \\ \nabla^2 A_y = -\mu_0 j_y \\ \nabla^2 A_z = -\mu_0 j_z \end{cases}$$

this is like Poisson eqn of electrostatics

$$\nabla^2\phi = -\rho/\epsilon_0$$

whose sol'n we know

$$\phi(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')}{|\vec{r}-\vec{r}'|}\, d^3r'$$

So we can relate $\vec{A}$ to $\vec{J}$ in Coulomb gauge

$$A_x(\vec{r}) = +\frac{\mu_0}{4\pi}\int \frac{j_x(r')}{|\vec{r}-\vec{r}'|}\, d^3r'$$

or putting together with y, z

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(r')}{|r-r'|}\, d^3r'$$

This is precisely eqn for $\vec{A}(\vec{r})$ we previously derived from Biot-Savart law!

4-2A

① $\left[\vec{\nabla}\times(\vec{\nabla}\times\vec{A})\right]_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(\vec{\nabla}\times\vec{A})_k$

$$= \epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{k\ell m}\frac{\partial}{\partial x_\ell}A_m$$

$$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_\ell}A_m$$

$$= \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}A_j - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}A_i$$

$$= \frac{\partial}{\partial x_i}(\vec{\nabla}\cdot\vec{A}) - \nabla^2 A_i$$

$\Rightarrow \quad \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

② $(\vec{\nabla}\times\vec{\nabla}\psi)_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\psi$

Rename $i \leftrightarrow j$

$$= \epsilon_{jik}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\psi$$

$$= -\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\psi$$

partial derivatives commute

$\Rightarrow \phi$

Physically measurable quantities are always indep. of choice of gauge

$$\oint_C \vec{A}' \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{A}') \cdot \hat{n}\, dA = \int \vec{B}' \cdot \hat{n}\, dA = \phi_B'$$

$$\downarrow$$

$$\oint (\vec{A} + \vec{\nabla}\psi) \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{A} + \underbrace{\vec{\nabla} \times \vec{\nabla}\psi}_{0}) \cdot \hat{n}\, dA = \phi_B$$

so $\phi_B' = \phi_B$ magnetic flux is gauge invariant quantity

Aside zero of energy is arbitrary $E' = E + c$

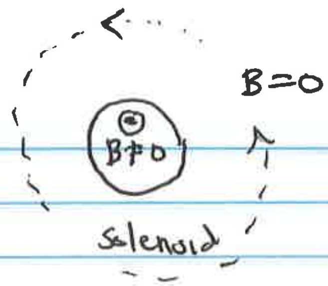
cannot measure energy but $C = \frac{dE}{dT} = \frac{dE'}{dT}$

is measurable and is independent of choice of c.

How to measure C: Add energy to system, known amount dE

and measure change in dT $\Rightarrow C$.

G-4



$$\oint \vec{A}\cdot d\vec{l} = \int \vec{\nabla}\times\vec{A}\cdot\hat{n}\,dA$$

$$= \int \vec{B}\cdot\hat{n}\,dA = \phi_B$$

Even though $\vec{B} = 0$ outside solenoid $\oint \vec{A}\cdot d\vec{l} \neq 0$

so $\vec{A}$ is <u>non zero</u> outside solenoid.