-1

HW 12.1.1, 12.1.2, 12.1.6
12.2.1, 12.2.2, 12.2.10

Legendre Functions

Generating function $\rightarrow$ $g(t,x) \equiv (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$

Two approaches to special functions in physics
① PDE separate variables etc find in solving get these special funcs
② "generating function"

Since 'rhs' is expansion in powers of t do so also for lhs

$$(1+u)^n = 1 + nu + \tfrac{1}{2}n(n-1)u^2 + \tfrac{1}{6}n(n-1)(n-2)u^3 + \dots$$

is binomial theorem

$$(1-2xt+t^2)^{-1/2} = 1 - \tfrac{1}{2}(-2xt+t^2)$$
$$+ \tfrac{1}{2}(-\tfrac{1}{2})(-\tfrac{3}{2})(-2xt+t^2)^2$$
$$+ \tfrac{1}{6}(-\tfrac{1}{2})(-\tfrac{3}{2})(-\tfrac{5}{2})(-2xt+t^2)^3$$

$$= 1 + xt - \tfrac{1}{2}t^2 + \tfrac{3}{8}(4x^2t^2 - 4xt^3 + t^4)$$
$$- \tfrac{5}{16}\left[-8x^3t^3 + 12x^2t^4 \dots\right]$$

$= t^0 \left[1\right]$ $\qquad P_0(x) = 1$

$+ t^1 \left[x\right]$ $\qquad P_1(x) = x$

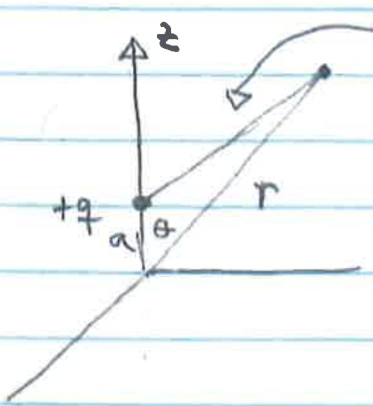
$+ t^2 \left[-\tfrac{1}{2} + \tfrac{3}{2}x^2\right]$ $\qquad P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2}$

$+ t^3 \left[-\tfrac{3}{2}x + \tfrac{5}{2}x^3\right]$ $\qquad P_3(x) = \tfrac{5}{2}x^3 - \tfrac{3}{2}x$

Where does this generating function come from physically?

One way to see it: compute $V(\vec{r})$ for charge q

a distance a from origin



$$V(r) = \frac{q}{4\pi\epsilon_0} \frac{1}{(r^2 + a^2 - 2ar\cos\theta)^{1/2}}$$

$$V(r) = \frac{q}{4\pi\epsilon_0 r} \frac{1}{\left(1 - \frac{2a}{r}\cos\theta + a^2/r^2\right)^{1/2}}$$

Drop

Define $\cos\theta = x \qquad a/r = t$

$$(1 - 2tx + t^2)^{-1/2}$$ our generating function

$$V(r) = \frac{q}{4\pi\epsilon_0 r} \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n$$

Virtue: separates out angular structure

from distance structure

As in QM $\quad \psi_{n\ell m}(\vec{r}) = Y_{\ell m}(\theta,\phi)\, R_n(r)$

Legendre poly also arises from a particular differential Eqn. To see this

$$g(t,x) = (1-2xt+t^2)^{-1/2} = \sum_0^\infty P_n(x)\,t^n$$

$$\partial g/\partial x = \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_0^\infty P_n'(x)\,t^n$$

First

$$(1-2xt+t^2)\sum_0^\infty P_n'(x)\,t^n - t\sum_0^\infty P_n(x)\,t^n = 0$$

Coefficients of each power of t set to zero

$$P_n'(x) - 2x P_{n-1}'(x) + P_{n-2}'(x) - P_{n-1}(x) = 0$$

$*$ $$P_{n+1}'(x) + P_{n-1}'(x) = 2x\,P_n'(x) + P_n(x)$$

① Also First

$$\partial g/\partial t = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_0^\infty n P_n(x)\,t^{n-1}$$

$$(1-2xt+t^2)\sum n P_n(x)\,t^{n-1} + (t-x)\sum P_n(x)\,t^n = 0$$

Same trick of identifying powers of t:

$$(n+1)P_{n+1}(x) = (2n+1)\,x P_n(x) - n P_{n-1}(x)$$

$n=1$ $\quad 2P_2(x) = 3x\,P_1(x) - P_0(x) = 3x^2 - 1$ ✓

$n=2$ $\quad 3P_3(x) = 5x\,P_2(x) - 2P_1(x) = 5x(3x^2-1)\tfrac{1}{2} - 2x$ ✓

This is called a "recursion relation".

Let's differentiate wrt x & $\times 2$

$$(n+1)P'_{n+1}(x) = (2n+1)P_n(x) + (2n+1)xP'_n(x) - nP'_{n-1}(x)$$

$$\cancel{(2n+1)\times * \quad (2n+1)P'_{n+1}(x) + (2n+1)P'_{n-1}(x)}$$

$$\cancel{= 2x(2n+1)P_n(x) + (2n+1)P_n(x)}$$

This is a second eqn involving P'_{n+1}, P_n, P'_n, P_{n-1}

Playing around with combining this with $*$ one

can ultimately show

(Proof in text)

$$(1-x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0$$

Price paid for getting eqn for just one $P_n(x)$ is that it is now <u>second</u> order

$$1-x^2\frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x} + (n+1)n$$

Is this a Hermitian differential operator? If so $P_n(x)$ are <u>complete</u>

Exercise

Talk about this more later in terms of our general theory of eigenfunctions of Hermitian operators

But first . . special values

set $x = 1$ in generating function

$$\frac{1}{(1-2t+t^2)^{1/2}} = \frac{1}{1-t} = \sum_0^{\infty} t^n$$

But also $\qquad = \sum P_n(1)\, t^n$

so $\qquad P_n(1) = 1$

$\qquad P_n(-1) = (-1)^n$ $\Bigg\}$ check it out

orthogonality $\qquad \int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \frac{2}{(2n+1)}\,\delta_{m,n}$

Check cases . . .

$$\int_{-1}^{1} P_1(x) P_3(x) dx = \int_{-1}^{1} x\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) dx$$

$$= \int_{-1}^{1} \left(\frac{5}{2}x^4 - \frac{3}{2}x^2\right) dx = \frac{1}{2}x^5 - \frac{1}{2}x^3\Big|_{-1}^{1} = 0$$

$$\int_{-1}^{1} P_2(x) P_2(x) dx = \int_{-1}^{1} \left(\frac{3}{2}x^2 - \frac{1}{2}\right)^2 dx$$

$$= \frac{1}{4}\int_{-1}^{1} (9x^4 - 6x^2 + 1)\, dx = \frac{1}{4}\left(\frac{9}{5}x^5 - 2x^3 + x\right)\Big|_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{9}{5} - 2 + 1\right) = \frac{1}{2}\,\frac{4}{5} = \frac{2}{5} = \frac{2}{2(2)+1}$$

proofs of orthogonality

(2) using generating function we get normalization

$$(1-2xt+t^2)^{-1} = \left(\sum_0^\infty P_n(x)t^n\right)^2$$

$$\int_{-1}^{1} dx\,(1-2xt+t^2)^{-1} = \sum_0^\infty t^{2n}\int_{-1}^{1}[P_n(x)]^2\,dx$$

$$y = 1-2tx+t^2 \qquad dy = -2t\,dx$$

$$\rightarrow \int_{(1+t)^2}^{(1-t)^2} \frac{dy/(-2t)}{y} = \frac{1}{2t}\ln y\Big|_{(1-t)^2}^{(1+t)^2} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right)$$

(1) using general theory from 204A Sturm-Liouville theory

$$\mathcal{L} = p_0(x)\,d^2/dx^2 + p_1(x)\,d/dx + p_2(x)$$

$\mathcal{L}$ is self-adjoint (aka Hermitian) if $p_1(x) = p_0'(x)$

$$\left[(1-x^2)\,d^2/dx^2 - 2x\,d/dx + n(n+1)\right]P_n(x) = 0$$

✓

$\therefore$ $\{P_n(x)\}$ are orthogonal

Indeed? $\displaystyle\int_a^b f\mathcal{L}g\,dx = \int_a^b g\mathcal{L}f\,dx + p_0(x)[gf'-g'f]\Big|_a^b$

$$p_0(a) = p_0(b) = 0 \qquad \begin{matrix} a=-1 \\ b=1 \end{matrix}$$

$$\ln(1+t) = t - t^2/2 + t^3/3 - t^4/4$$

$$\ln(1-t) = -t - t^2/2 - t^3/3 - t^4/4$$

$$\ln(1+t) = \int \frac{dt}{1+t} = \int (1 - t + t^2/2 - t^3/3 \ldots)\, dt$$

$$\ln[(1+t)/(1-t)] = \ln(1+t) - \ln(1-t)$$

$$= 2\left(t + t^3/3 + t^5/5 + \ldots\right)$$

$$\frac{1}{t}\ln\frac{1+t}{1-t} = 2\left(1 + \frac{t^2}{3} + \frac{t^4}{5} + \frac{t^6}{7} + \ldots\right) = \sum_0^\infty \frac{2}{2n+1} t^{2n}$$

$$= \sum_0^\infty \left[\int_{-1}^1 [P_n(x)]^2\, dx\right] t^{2n}$$

$$\therefore \int_{-1}^1 (P_n(x))^2\, dx = \frac{2}{2n+1}$$

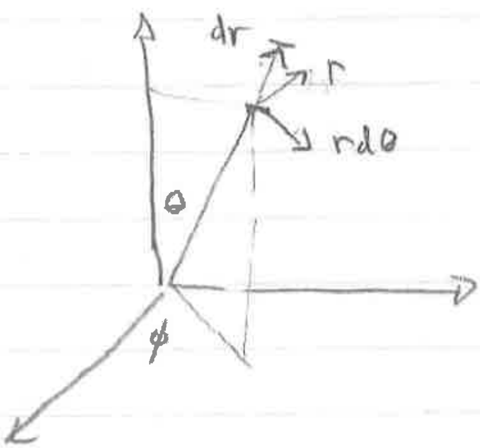
Applications of Legendre Poly's to EM

We have seen Coulomb's Law $\Rightarrow$ Legendre poly's from viewpoint of generating functions.

What about diff. eqn? $\nabla^2 V = 0$ if $\rho = 0$

What is ∇^2 in r, θ, ϕ coordinates

$$\vec{\nabla} V = \hat{r} 1 \frac{\partial V}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial V}{\partial \theta} + \hat{\phi} \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi}$$



$ds = 1 dr$ if r changed $r + dr$

$ds = r\, d\theta$ if θ changed $\theta + d\theta$

$ds = r \sin\theta\, d\phi$ if ϕ changed $\phi + d\phi$

General rule: if, curvilinear coordinates $q_1\ q_2\ q_3$ used instead of $x\ y\ z$

But these coordinates are still orthogonal (see above)

$$\nabla^2 V = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \frac{h_2 h_3}{h_1} \frac{\partial V}{\partial q_1} + \frac{\partial}{\partial q_2} \frac{h_1 h_3}{h_2} \frac{\partial V}{\partial q_2} + \frac{\partial}{\partial q_3} \frac{h_1 h_2}{h_3} \right.$$

~~Complex formula because $\hat{\theta}, \hat{\phi}, \hat{r}$ depend on~~

where $ds = \left.\begin{array}{l} h_1\, dq_1 \\ h_2\, dq_2 \\ h_3\, dq_3 \end{array}\right\}$ changing q_1, q_2, q_3 respectively

$r \quad h_1 = 1$

$\theta \quad h_2 = r$

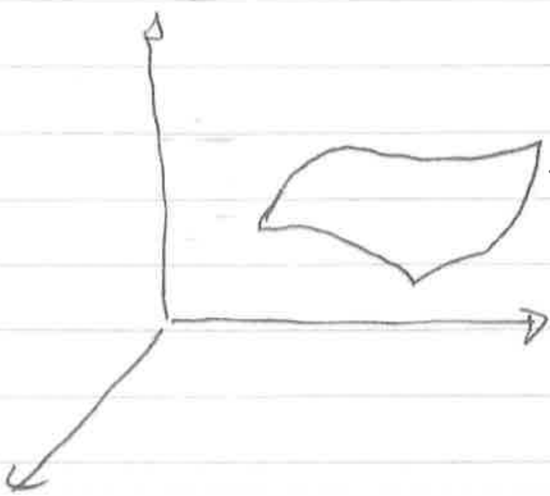
$\phi \quad h_3 = r\sin\theta$

$$\nabla^2 V = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r} r^2\sin\theta \frac{\partial V}{\partial r} + \frac{\partial}{\partial \theta}\sin\theta\frac{\partial V}{\partial \theta} + \frac{\partial}{\partial \phi}\frac{1}{\sin\theta}\frac{\partial V}{\partial \phi}\right]$$

why so complicated? $\hat{r}, \hat{\theta}, \hat{\phi}$ depend on r, θ, ϕ

as oppose to $\hat{x}\ \hat{y}\ \hat{z}$ being same everywhere in space

Differential geometry



"Manifold" a lower dimensional object embedded in a higher d space

How can you define unit vectors, derivatives on Manifold $\boxed{\text{w/o}}$ reference to the higher d space ?!

L-10 $\quad V(r,\theta,\phi) \to V(r,\theta) \quad \nabla^2 V = 0$ but be a bit more general

$$V(r\,\theta\,\phi) = R(r)\,T(\theta)\,S(\phi)$$

$$\frac{1}{r^2}\,TS\,\frac{d}{dr}r^2\frac{dR}{dr} + \frac{RS}{r^2\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dT}{d\theta} + \frac{RT}{r^2\sin^2\theta}\frac{d^2S}{d\phi^2}$$

$$= -k^2\,RTS$$

Helmholtz eqn

$$\frac{1}{S}\frac{d^2S}{d\phi^2} = r^2\sin^2\theta\left[-k^2 - \frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{1}{r^2\sin\theta\,T}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)\right]$$

rhs is indep of ϕ

m must be an integer?

$$\frac{1}{S}\frac{d^2S}{d\phi^2} = -m^2 \qquad S(\phi) = e^{\pm im\phi}$$

or $\cos m\phi$, $\sin m\phi$

$$\frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{r^2\sin\theta\,T}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) - \frac{m^2}{r^2\sin^2\theta} = -k^2$$

$$Q = \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + r^2k^2 = -\frac{1}{\sin\theta\,T}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) + \frac{m^2}{\sin^2\theta}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2R - \frac{QR}{r^2} = 0$$

$$+\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) - \frac{m^2}{\sin^2\theta}T + QT = 0$$

L-10A

$$D\,\nabla^2 \psi(r,t) = \frac{\partial \psi(r t)}{\partial t}$$

$$\psi = f(\bar{r})\, g(t)$$

$$D g\, \nabla^2 f = f\, \partial g/\partial t$$

$$\frac{\nabla^2 f}{f} = \frac{1}{D g}\, \partial g/\partial t = -k^2$$

$$g = e^{-Dk^2 t}$$

$$\nabla^2 f = -k^2 f \qquad e^{ikx}$$

1-d :

$$\psi(x t) = \int_{-\infty}^{\infty} a(k)\, e^{ikx}\, e^{-Dk^2 t}$$

$\psi(x,0)$ determines $a(k)$ etc.

Let's now set $k=0$ so we are considering the Laplace eqn instead of the more general Helmholtz eqn.

We will come back to Helmholtz where we will see radial eqn leads to Bessel functions. But $k^2=0$ is much simpler so focus on it for now.

Even so, there is a subtle point. This is the fact that Q cannot be arbitrary but must equal $l(l+1)$ where l is an integer!

Sort of analogous to restriction that m = integer so that $P(\phi+2\pi)=P(\phi)$

Let $x=\cos\theta$ $\quad T(\theta)\to T(x)$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dT}{d\theta}+\left(Q-\frac{m^2}{\sin^2\theta}\right)T=0$$

$$\frac{d}{d\theta}=\frac{dx}{d\theta}\frac{d}{dx}=-\sin\theta\frac{d}{dx} \qquad \sin\theta=\sqrt{1-x^2}$$

$$-\frac{1}{\sqrt{1-x^2}}\sqrt{1-x^2}\frac{d}{dx}\sqrt{1-x^2}\frac{dT}{dx}+\left(Q-\frac{m^2}{1-x^2}\right)T=0$$

$$-\left[\frac{d^2T}{dx^2}+\frac{2x}{\sqrt{1-x^2}}\frac{dT}{dx}\right]+\left(Q-\frac{m^2}{1-x^2}\right)T=0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)+\left(Q-\frac{m^2}{\sin^2\theta}\right)T=0$$

$$x=\cos\theta \qquad \sin\theta=\sqrt{1-x^2}$$

$$\frac{d}{d\theta}=\frac{dx}{d\theta}\frac{d}{dx}=-\sin\theta\frac{d}{dx}$$

$$+\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}T\right]+\left(Q-\frac{m^2}{1-x^2}\right)T=0$$

$$(1-x^2)\frac{d^2T}{dx^2}-2x\frac{dT}{dx}+\left(Q-\frac{m^2}{1-x^2}\right)T=0$$

Let's consider in detail the case when $Q=\ell(\ell+1)$.

Defining $P(x)=(1-x^2)^{-m/2}\,T(x)$ yields

$$* \qquad (1-x^2)\frac{d^2P}{dx^2}-2(m+1)x\frac{dP}{dx}+(\ell-m)(\ell+m+1)P=0$$

For $m=0$

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+\ell(\ell+1)P=0$$

$P=P_\ell=$ Legendre polynomials

L-12 A

$$T(x) = (1-x^2)^{m/2} p(x)$$

$$T'(x) = (1-x^2)^{m/2-1} \frac{m}{2}(-2x) P(x) + (1-x^2)^{m/2} p'(x)$$

$$T''(x) = (1-x^2)^{m/2-2} \left(\tfrac{m}{2}-1\right)(-mx) P(x)$$

$$+ (1-x^2)^{m/2-1} (-m) P(x)$$

$$+ (1-x^2)^{m/2-1} (-mx) p'(x)$$

$$+ (1-x^2)^{m/2-1} \frac{m}{2}(-2x) p'(x)$$

$$+ (1-x^2)^{m/2} p''(x)$$

$$(1-x^2) T'' - 2xT' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right] T = 0$$

$$(1-x^2)^2 T'' - 2x(1-x^2) T' + (\ell(\ell+1)(1-x^2) - m^2) T = 0$$

$$(1-x^2)^{m/2} \left(\tfrac{m}{2}-1\right)(-mx) p(x) \quad + (1-x^2)^{m/2}(-2x)(-mx) p(x)$$

$$+ (1-x^2)^{m/2} (1-x^2)(-m) p(x) \quad + (1-x^2)^{m/2} (-2x) p'(x)$$

$$+ (1-x^2)^{m/2} (1-x^2)(-mx) p'(x) \quad + [\ell(\ell+1)(1-x^2) - m^2](1-x^2)^{m/2} p(x)$$

$$+ (1-x^2)^{m/2} (1-x^2)(-mx) p'(x) \quad = 0$$

$$+ (1-x^2)^{m/2} (1-x^2) p''(x)$$

L-14B

cancelling $(1-x^2)^{m/2}$

$\cancel{(\frac{m}{2}-1)(-mx)}$

$$P(x)\left[\left(\frac{m}{2}-1\right)(-mx)+(1-x^2)(-m)-2x(-mx)+l(l+1)(1-x^2)-m^2\right]$$

$$P'(x)\left[2(1-x^2)(-mx)-2x\right]$$

$$P''(x)\left[(1-x^2)\right]$$

$\ell^2+\ell-2=(\ell-1)(\ell+2)$

Start with m=0 eqn

$$\frac{d}{dx}: \quad (1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \ell(\ell+1)P = 0$$

$$-2x\,P'' + (1-x^2)\,P''' - 2P' - 2xP'' + \ell(\ell+1)P' = 0$$

get m=1 eqn

$$(1-x^2)\frac{d^2}{dx^2}(P') - 2(1+1)x\frac{d}{dx}P' + (\ell-1)(\ell+1+1)P' = 0$$

$$(1-x^2)\frac{d^2}{dx^2}(P') - 2(m+1)x\frac{d}{dx}P' + (\ell-m)(\ell+m+1)P' = 0$$

~~If P_ℓ obeys m=0 eqn P_ℓ' obeys m=1 eqn~~

So our soln's for $m \neq 0$ are obtained from the m=0 solns by differentiation!

$P_\ell^{m=1}(x) = \frac{d}{dx}P_\ell^{m=0}(x)$

$$T(x) = \cancel{P_\ell^m(x)} = \cancel{(-1)^m}(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x)$$

Associated Legendre Functions

$T(x) = (1-x^2)^{m/2}P(x)$

are solns to the θ part of the Laplace eqn

$$S(\phi) = e^{\pm im\phi}$$

Finally, radial part

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{dR}{dr} + \cancel{k^2}R - \frac{\ell(\ell+1)R}{r^2} = 0$$

$$\frac{d}{dr}r^2\frac{dR}{dr} - \ell(\ell+1)R = 0$$

$$R = r^\ell \qquad \frac{dR}{dr} = \ell r^{\ell-1} \qquad r^2\frac{dR}{dr} = \ell r^{\ell+1}$$

$$R = r^{-(\ell+1)} \qquad \frac{dR}{dr} = -(\ell+1)r^{-(\ell+2)} \qquad r^2\frac{dR}{dr} = -(\ell+1)r^{-\ell}$$

L-14

Soln to Laplace Eqn in spherical coordinates

$R(r)\,T(\cos\theta)\,S(\phi)$

$R_\ell(r) = r^\ell$ or $r^{-(\ell+1)}$

$S_m(\phi) = e^{im\phi}$

$x = \cos\theta$ $\quad T(x) = P_\ell^m(x) = \cancel{(-1)^m}(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x)$

$(1-x^2)^{m/2}$

$P_\ell^0(x) = P_\ell(x)$

~~We never said why we forced $\alpha = \ell(\ell+1)$!~~

$P_1^0(x) = P_1(x) = x = \cos\theta$

$P_1^1(x) = (1-x^2)^{1/2}\frac{d}{dx}\underbrace{P_1^0(x)}_{x} = (1-x^2)^{1/2} = \sin\theta$

$P_2^0(x) = P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

$P_2^1(x) = \cancel{(-1)}(1-x^2)^{1/2}\frac{d}{dx}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$

$= \cancel{(-1)}(1-x^2)^{1/2}(3x)$

$= 3\cos\theta\sin\theta$

$P_2^2(x) = (1-x^2)^1\frac{d^2}{dx^2}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) = (1-x^2)^1 3$

$= 3\sin^2\theta$

L-15

One often combines θ, ϕ dependence into spherical Harmonics

$$Y_\ell^m(\theta,\phi) = (-1)^m \left[\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\cos\theta)\, e^{im\phi}$$

normalization factor

$$\int_0^{2\pi} d\phi \int_0^\pi \sin\theta\, d\theta \quad Y_\ell^m \; Y_{\ell'}^{m'} = \delta_{\ell\ell'}\delta_{mm'}$$

$$Y_0^0 = \frac{1}{2\sqrt{\pi}}$$

$$Y_1^0 = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$$

$$Y_1^{\pm 1} = \mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{\pm i\phi}$$

$$Y_2^0 = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(2\cos^2\theta - \sin^2\theta\right)$$

$$3x^2 - 1 = 3\cos^2\theta - 1\;(\cos^2\theta - \sin^2\theta)$$
$$= 2\cos^2\theta - \sin^2\theta$$

$$Y_2^{\pm 1} = \mp\frac{1}{2}\sqrt{\frac{15}{2\pi}}\cos\theta\sin\theta\, e^{\pm i\phi}$$

$$Y_2^{\pm 2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{\pm 2i\phi}$$

$r^\ell \qquad r^{-(\ell+1)}$

Summary

To solve Laplace's Eqn, take linear combinations of

$$V(r,\theta,\phi) = \sum \quad r^{\ell} Y_{\ell}^{m}(\theta,\phi) \qquad r^{-(\ell+1)} Y_{\ell}^{m}(\theta,\phi)$$

coefficients will be chosen to fit boundary conditions

To solve Helmholtz Eqn : r^{ℓ} $r^{-(\ell+1)}$

get more complex (Bessel Eqns)

1. Spherical conductor radius R divided into 2 hemispheres by thin layer of insulating material. upper at u_1 lower u_2. Find ϕ outside sphere

$$V(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + B_n r^{-(n+1)}\right)P_n(\cos\theta)$$

clearly $A_n = 0 \ \forall n$

$$V(r,\theta) = \sum_n B_n r^{-(n+1)} P_n(\cos\theta)$$

we have $V(R,\theta) = \sum_n B_n R^{-(n+1)} P_n(\cos\theta) = \begin{cases} u_1 & 0 \le \theta \le \pi/2 \\ u_2 & \frac{\pi}{2} \le \theta \le \pi \end{cases}$

Now $\int_0^\pi \sin\theta\, d\theta\, P_n(\cos\theta) P_m(\cos\theta) = \frac{2}{2n+1}\delta_{nm}$

So $\int_0^\pi \sin\theta\, d\theta\, V(R,\theta) P_m(\cos\theta) = \frac{2}{2m+1} R^{-(m+1)} B_m$

$$B_m = \frac{2m+1}{2R^{m+1}}\left\{\int_0^{\pi/2} \sin\theta\, d\theta\, u_1 P_m(\cos\theta) + \int_{\pi/2}^{\pi} \sin\theta\, d\theta\, u_2 P_m(\cos\theta)\right.$$

$$= \frac{2m+1}{2R^{m+1}}\left\{\int_0^\pi \sin\theta\, d\theta\, \frac{u_1+u_2}{2} P_m(\cos\theta)\, d\theta + \int_0^{\pi/2} \sin\theta\, d\theta\, \frac{u_1-u_2}{2} P_m(\cos\theta)\right.$$

$$+ \int_{\pi/2}^{\pi} \sin\theta\, d\theta\, \frac{u_2-u_1}{2} P_m(\cos\theta)$$

Now $\int_0^\pi \sin\theta\, d\theta\, \frac{u_1+u_2}{2} P_m(\cos\theta) = \frac{u_1+u_2}{2}\,\frac{2}{1}\,\delta_{m,0}$

$\frac{u_1-u_2}{2}\left[\int_0^{\pi/2} d\theta \sin\theta\, P_m(\cos\theta) - \int_{\pi/2}^{\pi} d\theta \sin\theta\, P_m(\cos\theta)\right]$ nonzero only for odd m

Anyway these determine B_m and

$$V(r,\theta) = \sum_m B_m r^{-(m+1)} P_m(\cos\theta)$$

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2}{2n+1} \delta_{nm}$$

$x = \cos\theta \qquad dx = -\sin\theta\, d\theta$

$$\int_{\pi}^{0} P_n(\cos\theta) P_m(\cos\theta)(-\sin\theta\, d\theta) = \frac{2}{2n+1} \delta_{nm}$$

$$\int_{0}^{\pi} d\theta \sin\theta\, P_n(\cos\theta) P_m(\cos\theta) = \frac{2}{2n+1} \delta_{nm}$$

We have $\displaystyle\int_0^{\pi/2} u_1 P_n(\cos\theta)\, \underset{\wedge}{\overset{\sin\theta}{d\theta}} + \int_{\pi/2}^{\pi} u_2 P_n(\cos\theta)\, \underset{\wedge}{\overset{\sin\theta}{d\theta}}$

$x = \cos\theta$

$$= u_1 \int_1^0 P_n(x)\, dx\,(-1) + \int_0^{-1} u_2 P_n(x)(-1)\, dx$$

$$= u_1 \int_0^1 P_n(x)\, dx + u_2 \int_{-1}^{0} P_n(x)\, dx$$

$n = 0 \quad P_0(x) = 1 \qquad u_1 + u_2$

$n = 1 \quad P_1(x) = x \qquad \frac{1}{2}(u_1 - u_2)$

$\int \to x^2/2$

0

$n = 2 \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

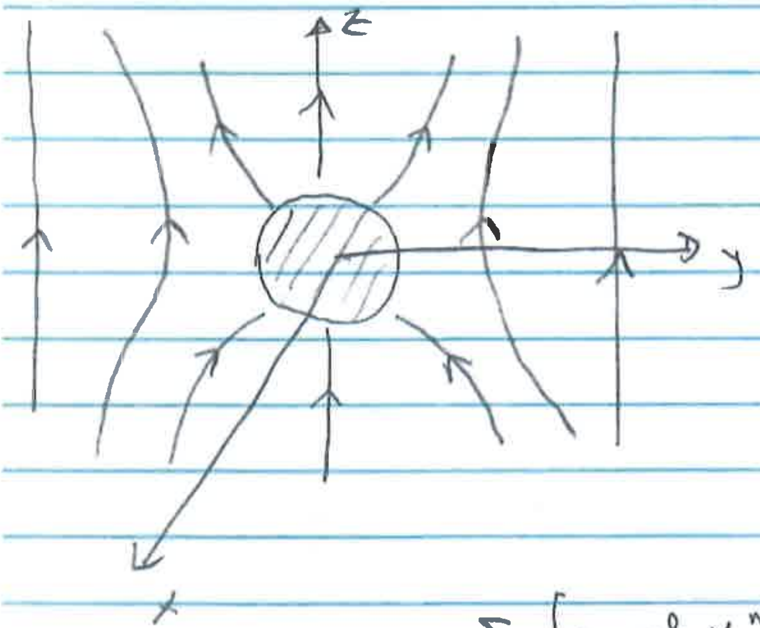
$\int \to x^3/2 - x/2$

$n = 3 \quad \frac{5}{2}x^3 - \frac{3}{2}x^2$

$\int \to$

L-17

conducting sphere in a uniform E field



No ϕ dependence by symmetry

$$\nabla^2 V = 0$$

$$V = \sum_{\ell,m}\left[a_{\ell m} r^{\ell} Y_\ell^m(\theta,\phi) + b_{\ell m} r^{-(\ell+1)} Y_\ell^m(\theta,\phi)\right]$$

No ϕ dependence : ~~use only~~ $m=0$ P_ℓ the Legendre funct

$$= \sum_\ell a_\ell r^\ell P_\ell(\cos\theta) + b_\ell r^{-(\ell+1)} P_\ell(\cos\theta)$$

$r \to \infty$ $\quad \vec{E} = E_0 \hat{z} = -\vec{\nabla} V$

$$V = -E_0 z = -E_0 r\cos\theta = -E_0 r P_1(\cos\theta)$$

$$a_\ell = 0 \quad \ell > 1 \qquad a_1 = -E_0$$

L-18

$r = r_0$

∵ sphere is an equipotential, choose it to be $V=0$

$$-E_0 r_0 P_1(\cos\theta) + \sum_\ell b_\ell r_0^{-(\ell+1)} P_\ell(\cos\theta) = 0$$

But the P_ℓ are linearly independent so $b_\ell = 0$ except for b_1 and, in fact

$$-E_0 r_0 + b_1 r_0^{-2} = 0$$

$$b_1 = E_0 r_0^3$$

$$V = \left[-E_0 r + \frac{E_0 r_0^3}{r^2}\right] P_1(\cos\theta)$$

$\boxed{\text{Compute charge distribution on sphere}}$

Gauss' law $\quad \phi = q/\epsilon_0$

~~$E_{normal} A = \sigma A/\epsilon_0$~~

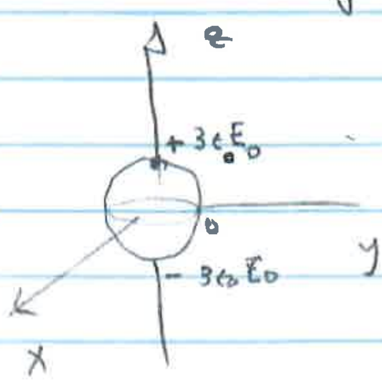
$$-\frac{\partial V}{\partial r} = \sigma/\epsilon_0$$

$P_1(\cos\theta)$

↓

$$\sigma = -\epsilon_0 \left.\partial V/\partial r\right|_{r=r_0} = -\epsilon_0 (-E_0 - 2E_0) \cos\theta$$

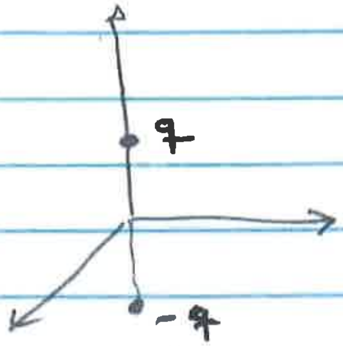
$$\sigma = 3\epsilon_0 E_0 \cos\theta$$



L-19

The expression $\frac{E_0 r_0^3}{r^2}\cos\theta$

is precisely the potential due to an electric dipole



$$V_+(r) = \frac{q}{4\pi\epsilon_0}\frac{1}{(r^2+a^2-2ar\cos\theta)^{1/2}}$$

generating funct

$$= \frac{q}{4\pi\epsilon_0 r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n$$

$$V_-(r) = \frac{-q}{4\pi\epsilon_0 r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n$$

$n=0$ cancels

$$n=1 \quad V(r) = \frac{+q}{4\pi\epsilon_0 r}P_1(\cos\theta)\left(\frac{a}{r}\right)2$$

$$V(r) = \frac{2q}{4\pi\epsilon_0}\frac{a}{r^2}\cos\theta = \frac{p}{4\pi\epsilon_0 r^2}\cos\theta$$

we have $E_0 r_0^3\frac{\cos\theta}{r^2}$

leading us to identify $\frac{p}{4\pi\epsilon_0} = E_0 r_0^3$

$$\boxed{p = 4\pi r_0^3\epsilon_0 E_0}$$

completeness of spherical harmonics

$$f(\theta,\phi) = \sum_{m\,n} a_{\ell m}\, Y_\ell^m(\theta,\phi)$$

$$a_{\ell m} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, Y_\ell^{m*}(\theta,\phi)\, f(\theta,\phi)$$

because $\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, Y_\ell^{m*}(\theta,\phi)\, Y_{\ell'}^{m'}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}$

If there is no ϕ dependence

$$f(\theta) = \sum_n a_\ell Y_\ell^0(\theta,\phi)$$

$$Y_\ell^0(\theta,\phi) = P_\ell^0(\cos\theta)\, \cancel{e^{i0\phi}}$$

why is $Q = \ell(\ell+1)$

$$\left[(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + Q - \frac{m^2}{1-x^2}\right]T(x) = 0$$

$*$ $\begin{cases} \mathcal{L}T(x) = QT(x) \\ \text{with} \quad \mathcal{L} = (1-x^2)\frac{d}{dx^2} - 2x\frac{d}{dx} - \frac{m^2}{1-x^2} \end{cases}$ ← Hermitial

In general we have T_{QM} $\perp$ & complete

$**$ $$\int_{-1}^{1} dx\, T_{QM}(x) T_{Q'm}(x) = \delta_{QQ'}$$

Consider $Q \neq \ell(\ell+1)$

$$T_{QM}(\cos\theta)e^{im'\phi} = \sum_{\ell m} a_{\ell m} Y_\ell^m(\theta,\phi)$$

$$a_{\ell m} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, Y_\ell^{m*}(\theta,\phi)\, T_{QM}(\cos\theta)\, e^{im'\phi}$$

$\sim e^{-im'\phi}$ so $m = m'$

$Y_\ell^m(\theta,\phi)$ is soln of $*$

so by $**$ integral gives zero

General Discussion / Background
"Special" Functions

We have been using $\left\{ \sin\frac{n\pi x}{L}, \cos\frac{n\pi x}{L} \right\}$ FOURIER

are complete, orthogonal set of functions on $[0, L]$

Any $^{(+)}$ $f(x) = \sum_n a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}$

(*) $a_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx \qquad b_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L} dx$

(*) follows from $\int_0^L \sin\frac{n\pi x}{L} \sin\frac{n'\pi x}{L} dx = \frac{L}{2}\delta_{nn'}$

orthogonality

(NB commonly $\sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}$ normalized)

$\boxed{Q}$ ~~What~~ Is there some general principle behind all this?

$\boxed{Q}$ are there other collections of functions that work

Hint
Complete orthogonal set ← where encountered before?

SFG-2

Answer: Hermitian Matrices / operators

Discrete / Finite

Continuous / Infinite

$\vec{V} = \sum_i v_i \hat{e}_i$

components

$v_i = \hat{e}_i \cdot \vec{V}$

$|\psi\rangle = \int dx\, \psi(x) |x\rangle$

In QM parlance $\psi(x)$ are components of $|\psi\rangle$ in position basis

$\psi(x) = \langle x|\psi\rangle$

$\vec{v} \cdot \vec{w} = \sum v_i^* w_i$

$\langle \phi|\psi\rangle = \int dx\, \phi^*(x)\psi(x)$

$M_{ij} = M_{ji}^*$

$(0010000)\, M \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = M_{35} \rightsquigarrow$

$\int dx\, \phi^*(x) \mathcal{L}\, \psi(x)$

$= \int dx\, \psi^*(x) \mathcal{L}\, \phi(x)$

Check out $\mathcal{L} = d^2/dx^2$

$\boxed{Q}$

$$\int_a^b \phi^*(x) \frac{d^2}{dx^2} \psi(x)\, dx = \underbrace{\phi^*(x) \frac{d}{dx} \psi(x) \Big|_a^b}_{\text{vanish at } a,b} - \int_a^b \frac{d\phi^*}{dx} \frac{d\psi}{dx} dx$$

vanish at a,b or periodic functions → difference vanishes

$$= \underbrace{- \frac{d\phi^*}{dx} \psi(x) \Big|_a^b}_{\text{vanishes again}} + \int_a^b \frac{d^2\phi^*(x)}{dx^2} \psi(x)\, dx = \left[\int_a^b \psi^*(x) \frac{d^2}{dx^2} \phi(x) \right]^*$$

vanishes again

$\frac{d^2}{dx^2}$ is Hermitian $\boxed{A}$

SFG-3

Eigenfunctions of $\frac{d^2}{dx^2}$ are orthogonal + complete

$\Rightarrow$ Fourier !!

Natural Question: what other $\mathcal{L}$ are Hermitian

There are many and for particular ones

Legendre $\qquad \mathcal{L} = p_2(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_0(x)$

Bessel $\qquad$ Condition on p_2, p_1, p_0

Laguerre $\qquad$ for Hermitian?

One more comment $\quad d^2/dx^2\, \psi(x) = -\psi(x)$] Calculus problem

Sol'n $\rightarrow \sin x;\ \cos x$

$$\boxed{\frac{dp_2}{dx} = p_1}$$

but it turns out you have known these a long long time (encountered in geometry in HS) so no big deal you know what they look like properties like $\sin^2 x + \cos^2 x = 1$ etc etc

as this discussion has emphasize Legendre's really no different except you were not introduced to them in H.S. so you need to learn all their properties...

SFG-4

$$\frac{dp_2}{dx} = p_1(x)$$

More generally

$$p_2(x)\frac{d^2}{dx^2}\psi(x) + p_1(x)\frac{d}{dx}\psi(x) + p_0(x)\psi(x) = \lambda\, w(x)\psi(x)$$

weight function

	p_2	p_1	λ	w
Legendre	$1-x^2$	0	$l(l+1)$	1
Chebyshev	$(1-x^2)^{1/2}$	0	n^2	$(1-x^2)^{1/2}$
Bessel	x	$-n^2/x$	a^2	x
Laguerre	xe^{-x}	0	α	e^{-x}
Hermite	e^{-x^2}	0	2α	e^{-x^2}
SHO	1	0	ω^2	1