$\boxed{1}$

$$\nabla^2 \psi(x,y) = 0 \qquad \psi = f(x)g(y)$$

$\psi_s(x) = \psi(x, y=0)$ known

$$f''g + fg'' = 0$$

$$f''/f = -g''/g = -k^2$$

$$f(x) = e^{ikx} \qquad g(y) = e^{\pm ky}$$

Assume looking for soln with $\psi(x, y \to \infty) = 0$

Then get e^{-ky} soln only

$$\psi(x,y) = \int_{-\infty}^{\infty} dk\, e^{ikx} e^{-|k|y} a(k)$$

$$\psi_s(x) = \psi(x, y=0) = \int_{-\infty}^{\infty} dk\, e^{ikx} a(k)$$

$$a(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx'\, e^{-ikx'} \psi_s(x')$$

$$\psi(x,y) = \int_{-\infty}^{\infty} dx'\, \psi_s(x') \underbrace{\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} e^{-|k|y} e^{-ikx'}}_{g(x,x',y)}$$

$\boxed{2}$

$$\int_0^\infty \frac{dk}{2\pi} e^{-ky} e^{ik(x-x')} + \int_{-\infty}^0 e^{ky} e^{ik(x-x')} \frac{dk}{2\pi}$$

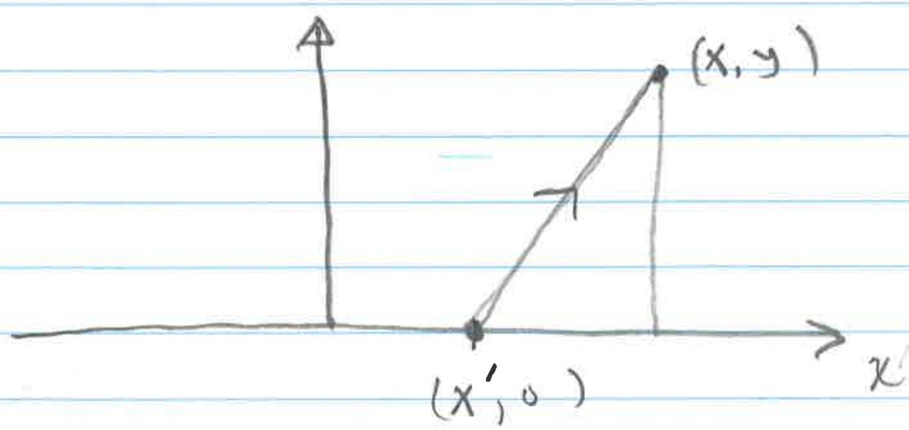
$$\frac{1}{2\pi} \quad \frac{e^{-ky+ik(x-x')}}{-y+i(x-x')} \Bigg|_0^\infty + \frac{e^{+ky+ik(x-x')}}{+y+i(x-x')} \Bigg|_{-\infty}^0$$

$$\frac{1}{2\pi} = \frac{-1}{-y+i(x-x')} + \frac{1}{y+i(x-x')}$$

$$\frac{1}{2\pi} = \frac{1}{y-i(x-x')} + \frac{1}{y+i(x-x')}$$

$$= \frac{y+i(x-x') + y-i(x-x')}{y^2+(x-x')^2}$$

$$\frac{1}{2\pi} = \frac{2y}{y^2+(x-x')^2}\left(\frac{1}{2\pi}\right)$$

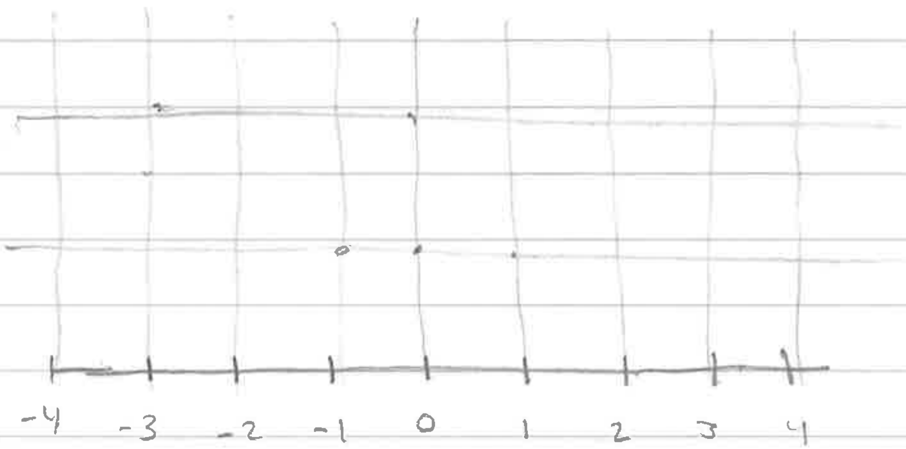


Value of potential at (x', 0) influences value at x, y

$$\star \quad \psi(x,y) = \int_{-\infty}^{\infty} dx' \, \psi_s(x') \frac{y}{\pi\left[y^2+(x-x')^2\right]}$$

Should $\psi(x,y) \to 0$ as $y \to \infty$?

Numerical soln



$$\nabla^2 \psi = -4\pi\rho$$

$$-4\pi \rho_{nm} = \frac{\psi(n,m+1) - 2\psi(n,m) + \psi(n,m-1)}{h^2} + \frac{\psi(n+1,m) - 2\psi(n,m) + \psi(n-1,m)}{h^2}$$

$$\psi(n,m) = \frac{1}{4}\left[\psi(n,m+1) + \psi(n,m-1) + \psi(n+1,m) + \psi(n-1,m)\right]$$

$$+ \frac{1}{4}\left(-4\pi\rho(n,m)\right)h^2$$

$$\parallel$$

$$0$$

$\boxed{4}$

$p(-1,1) = \frac{1}{4}\, p(-1,0)$

$p(0,1) = \frac{1}{4}\, p(0,0)$ numerical soln

$p(+1,1) = \frac{1}{4}\, p(1,0)$

$\nabla^2 \psi = 0 \qquad \psi_s(x) = \text{const} = V_0$

$\psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx'\, V_0 \; 2y/y^2 + (x-x')^2 \qquad \xi = x' - x$

$= \frac{y V_0}{\pi} \int_{-\infty}^{\infty} d\xi \; 1/y^2 + \xi^2 \qquad \xi = y \tan\theta$

$d\xi = y \sec^2\theta\, d\theta$

$= \frac{y V_0}{\pi} \int_{-\pi/2}^{\pi/2} \frac{y \cancel{\sec^2\theta}\, d\theta}{y^2 \cancel{\sec^2\theta}}$

$= \frac{V_0}{\pi}\, \pi \quad = V_0$

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Green's function is a way of writing general soln

for $\psi(x,y)$ in terms of values on boundary $[\psi_s(x')$ fore$]$

If $\psi_s(x') = \delta(x')$ $\quad \psi(x,y) = G(x,y)$

16.

Green's Functions combine pde + bc

Diffusion Eqn

$$D \frac{d^2 f}{dx^2} = \frac{\partial f}{\partial t} \qquad f = g(x)h(t)$$

$$D\, g''/g = h'/h = -k^2 D$$

$$g = e^{\pm ikx} \qquad h = e^{-Dk^2 t}$$

$$f(x,t) = \int_{-\infty}^{\infty} a(k)\, e^{-ikx}\, e^{-Dk^2 t}\, dk$$

$$f(x,0) = \int_{-\infty}^{\infty} a(k)\, e^{-ikx}\, dk$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{+ikx} f(x,0) = a(k)$$

$$f(x,t) = \int_{-\infty}^{\infty} dk\, e^{-Dk^2 t}\, e^{-ikx} \frac{1}{2\pi}\int dx'\, e^{ikx'} f(x',0)$$

$$= \int_{-\infty}^{\infty} dx'\, f(x',0) \underbrace{\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-ik(x-x')}\, e^{-Dk^2 t}}_{G(x-x',t)}$$

N.B. $f(x',0) = \delta(x')$

$f(x,t) = G(x,t)$

"Greens function is soln of pde with $\delta(x)$ input"

can do integral by completing square

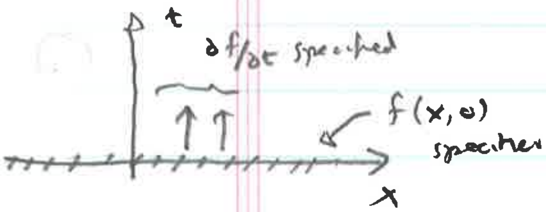
$$-Dk^2 t - ik(x-x') = -Dt\left(k^2 - \frac{ik(x-x')}{2Dt}\, 2\right) = \rightarrow$$

$$= -Dt\left(k - \frac{i(x-x')}{2Dt}\right)^2 - Dt\,\frac{(x-x')^2}{4D^2t^2} = -\frac{(x-x')^2}{4Dt}$$

16A

??

Suppose Neumann bc So that $\frac{\partial f}{\partial t}$ specified



$$\frac{\partial f}{\partial t} = \int_{-\infty}^{\infty} a(k)\, e^{-ikx} (-Dk^2)\, e^{-Dk^2 t}\, dk$$

$$\left.\frac{\partial f}{\partial t}\right|_{t=0} = \int_{-\infty}^{\infty} a(k)\, e^{-ikx} (-Dk^2)\, dk$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx'} \frac{\partial f}{\partial t}(x',0)\, dx' = -a(k)\, k^2 D$$

$$a(k) = \frac{-1}{2\pi D k^2}\int_{-\infty}^{\infty} dx'\, e^{ikx'} \frac{\partial f}{\partial t}(x',0)$$

$$f(x,t) = \int_{-\infty}^{\infty} dk\, e^{-Dk^2 t}\, e^{-ikx}\left(\frac{-1}{2\pi D k^2}\right)\int_{-\infty}^{\infty} dx'\, e^{ikx'} \frac{\partial f}{\partial t}(x',0)$$

$$= \int_{-\infty}^{\infty} dx' \frac{\partial f}{\partial t}(x',0) \underbrace{\int_{-\infty}^{\infty} \frac{-dk}{2\pi D k^2}\, e^{-ik(x-x')}\, e^{-Dk^2 t}}_{\tilde{G}(x-x',t)}$$

Does $\tilde{G}$ converge? Problem @ $k=0$?

(17)

Returning to Laplace, interesting to get $\psi(x,y)$ for upper half plane

First consider $b \to \infty$

$$\frac{\sinh \frac{n\pi}{a}(b-y)}{\sinh \frac{n\pi b}{a}} \to \frac{e^{n\pi/a(b-y)}\ \frac{1}{2}}{e^{n\pi/ab}\ \ 1/2} = e^{-n\pi y/a}$$

Then consider $a \to \infty$ and let Fourier sum $\to$ Fourier integral

$$\psi(x,y) = \sum_1^\infty \frac{2}{a}\int dx' \sin\frac{n\pi x'}{a}\psi_s(x') \sin\frac{n\pi x}{a} \frac{\sinh \frac{n\pi}{a}(b-y)}{\sinh n\pi b/a}$$

$\int_0^a dx' \to \int_0^\infty dx'$

$\int_{-\infty}^0 dx' + \int_0^\infty dx'$ these 2 pieces $\left\{ \left[\cos\frac{n\pi}{a}(x+x') - \cos\frac{n\pi}{a}(x-x')\right]\right.$

$\downarrow$

$e^{-n\pi y/a}$

$\downarrow$

e^{-ky}

$$\psi(x,y) = \frac{1}{\pi}\int_0^\infty e^{-ky}\,dy \int_{-\infty}^\infty dx'\,\psi_s(x')\cos k(x-x')$$

$$= \int_{-\infty}^\infty dx'\,\psi_s(x') \underbrace{\frac{1}{\pi}\int_0^\infty dk\, e^{-ky}\cos k(x-x')}_{G(x-x',y)}$$

Again can do integral

$$\frac{1}{2\pi}\int_0^\infty dk\, e^{-ky}\left(e^{ik(x-x')} + e^{-ik(x-x')}\right) = \frac{1}{2\pi}\left[\frac{e^{k(-y+i(x-x'))}}{-y+i(x-x')} + \frac{e^{k(-y-i(x-x'))}}{-y-i(x-x')}\right]_0^\infty$$

$$= \frac{1}{2\pi}\cancel{}\left[\frac{1}{y-i(x-x')} + \frac{1}{y+i(x-x')}\right]$$

$$= \frac{1}{2\pi}\frac{y+i(x-x') + y-i(x-x')}{y^2+(x-x')^2} = \frac{y}{y^2+(x-x')^2}\frac{1}{\pi}$$

17 A

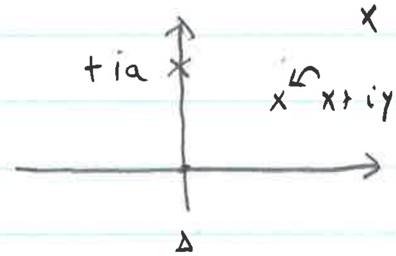
Hw problem

$$\psi(x,y) = \int_{-\infty}^{\infty} dx'\, \psi_s(x') \frac{y}{y^2+(x-x')^2}\frac{1}{\pi}$$

$$\psi_s(x') = \frac{1}{a^2+x'^2}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi}\, dx' \frac{1}{a^2+x'^2}\frac{y}{y^2+(x-x')^2}$$

x' plane



poles @ $x' = \pm ia$

$x'-x = \pm iy$

Looks like simple complex integration problem

$$\int_{-\infty}^{\infty} dx/(a^2+x^2)(b^2+x^2) = \cancel{2}\pi\cancel{i}\left[\frac{1}{\cancel{2i}a}\frac{1}{b^2-a^2} + \frac{1}{\cancel{2i}b}\frac{1}{a^2-b^2}\right]$$

$$1/(a^2+x^2)(b^2+x^2) = 1/(x+ia)(x-ia)(x^2+b^2) = 1/(x-ia)(x-ia+2ia)(\quad)$$

$$(x-ia)^2 = x^2 - 2iax - a^2 = x^2 - 2ia(x-ia) - a^2 + 2a^2$$

$$x^2 = (x-ia)^2 + 2ia(x-ia) + a^2$$

$$x^2+b^2 = (a^2+b^2) + 2ia(x-ia) + (x-ia)^2$$

$$\frac{1}{x^2+b^2} = \frac{1}{(b^2-a^2)}\left[1+\frac{2ia}{b^2-a^2}(x-ia) + \frac{(x-ia)^2}{b^2-a^2}\right]^{-1}$$

$$\frac{1}{x+ia} = \frac{1}{2ia}\left[1+\frac{x-ia}{2ia}\right]^{-1}$$

Could develop whole Laurent, but Residue is simply value at $x = ia$

$$\frac{1}{2ia}\frac{1}{b^2-a^2}$$

$$\int_{-\infty}^{+\infty} dx/(a^2+x^2)(b^2+x^2) = \frac{\pi}{b^2-a^2}\left[1/a - 1/b\right] = \frac{\pi}{(b^2-a^2)(ba)}(b-a)$$

$$= \pi/ab(a+b) \qquad \text{symmetric in } a/b$$

17A

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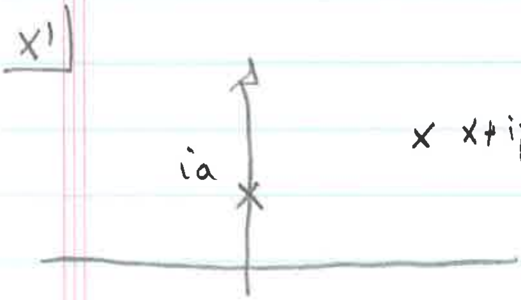
HW problem

$$\psi(x,y) = \int_{-\infty}^{\infty} dx'\, \psi_s(x')\, \frac{y}{y^2+(x-x')^2}\, \frac{1}{\pi}$$

$$\psi_s(x') = \frac{1}{a^2+x'^2}$$

$$\psi(x,y) = \int_{-\infty}^{\infty} dx'\, \frac{1}{a^2+x'^2}\, \frac{y}{y^2+(x-x')^2}\, \frac{1}{\pi}$$



$$\frac{1}{(x'+ia)(x'-ia)} \qquad \frac{1}{(x'-(x+iy))(x'-(x-iy))}$$

$x' = ia$ Residue $\displaystyle \frac{1}{2ia}\, \frac{1}{ia-x-iy}\, \frac{1}{ia-x+iy}$

$$= \frac{1}{2ia}\, \frac{1}{y^2+(x-ia)^2}$$

$x' = x+iy$ Residue $\displaystyle = \frac{1}{a^2+(x+iy)^2}\, \frac{1}{2iy}$

$$\text{Integral} = 2\pi i\, \frac{1}{\pi} \left\{ \frac{1}{2ia}\, \frac{1}{y^2+(x-ia)^2} + \frac{1}{2iy}\, \frac{1}{a^2+(x+iy)^2} \right\}$$

$$= \frac{1}{a}\, \frac{1}{y^2+x^2-2iax-a^2} + \frac{1}{y}\, \frac{1}{a^2+x^2-y^2+2ixy} \}$$

Real ??

(18)

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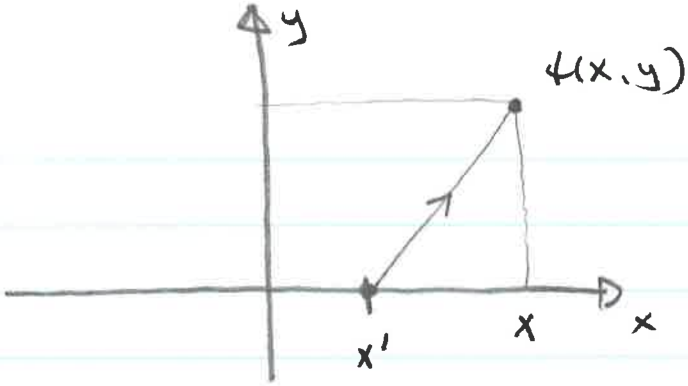


influence $\sim \dfrac{y}{y^2 + (x - x')^2}$

Time to emphasize two, closely related, types of Green's functions.

now this → ② inhomogeneous linear pde $\mathcal{L}\,\psi(\vec{r}) = \rho(\vec{r})$

solve for $\rho \sim \delta(r - r_0)$

$$\mathcal{L}\, G(r, r_0) = \delta(r - r_0)$$

$$\psi(r) = \int \rho(r_0)\, G(r, r_0)\, dr_0$$

we have done → ① homogenous pde given $\psi_s(r)$ or gradient on bdy

$G(r, r_0)$ is soln for $\psi_s(r) = 0$ everywhere on bdy <u>except at r_0</u>,

Full soln integrates up for all r_0.

* All this is not surprising. What is surprising is that these G are essentially the same!!

Language

inhomogeneous pde $\quad \nabla^2 \psi = -4\pi \rho \quad$ (Poisson)

homogeneous pde $\quad \nabla^2 \psi = 0 \quad$ (Laplace)

homogeneous bc $\quad \psi = 0$ on surface $\quad$ (or $\partial\psi/\partial n$)

inhomogeneous bc $\quad \psi \neq 0$ " "

Example of Relationship :

$$\nabla^2 G = -4\pi\, \delta(r - r_0)$$

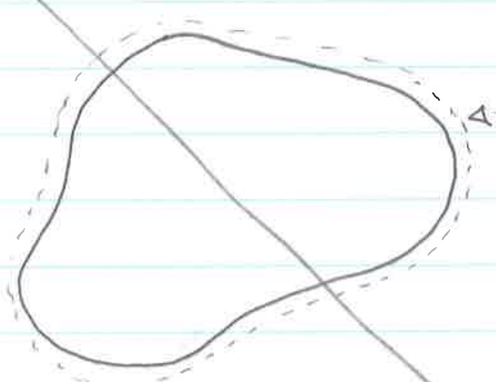
source

G: for inhomogeneous pde

has soln $\quad G = \dfrac{1}{|r - r_0|}$

How is it same G can be used to get soln to $\nabla^2 \psi = 0$ with $\psi_{surface}$ specified.

Basic idea



replace specified ψ_s by some charge distribution $\psi_s/4\pi$ distance ϵ away from S and $\psi_s = 0$

$\psi = -\dfrac{4\pi\sigma}{\epsilon}\, x \quad$ at $x = \epsilon \Rightarrow -4\pi\sigma$

gauss' law for E

Proof by example

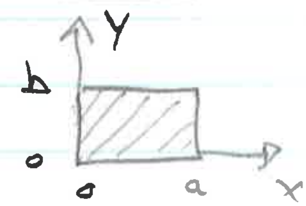
We solved $\nabla^2\psi = 0$ subject to $\psi_s(x) = \psi(x, y=0)$

$$\boxed{G(x; x', y) = \sum_{n=1}^{\infty} \frac{2}{a} \sin\frac{n\pi x'}{a} \sin\frac{n\pi x}{a} \frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a}}$$

no y' because we specified $\psi_s(x)$ at $y'=0$

$$\frac{\sinh \frac{n\pi}{a}(b-y+y')}{\sinh\frac{n\pi b}{a}} \to$$

Let's examine $\nabla^2\psi = -4\pi \rho(x,y)$ $\quad \psi = 0$ on bdy



Expand $\rho(x,y) = \sum_{nm} P_{nm} \sin\frac{m\pi x}{a} \sin\frac{m\pi y}{b}$

$$\psi(x,y) = \sum_{nm} A_{nm}$$

$$\nabla^2\psi = \sum_{nm}\left(-A_{nm}\left(\frac{m^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right)\right) \sin\frac{m\pi x}{a} \sin\frac{m\pi y}{b}$$

$$A_{nm}\left(\frac{m^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2 = 4\pi P_{nm}$$

$$\psi = \sum_{nm} \frac{4}{\pi} \frac{P_{nm}}{(m/a)^2 + (m/b)^2} \sin\frac{m\pi x}{a} \sin\frac{m\pi y}{b}$$

Since $P_{nm} = \int_0^a dy' dx' \left[\rho(x',y') \sin\frac{m\pi x'}{a} \sin\frac{m\pi y'}{b}\right] \frac{2}{a}\frac{2}{b}$

$$\psi(x,y) = \int_0^a dx' \int_0^b dy' \, \rho(x'y') \, G(x,x';y,y')$$

$$\boxed{G(xx'yy') = \sum_{nm} \frac{4}{\pi} \frac{4}{ab} \frac{\sin m\pi x/a \, \sin m\pi x'/a \, \sin m\pi y/b \, \sin m\pi y'/b}{(m/a)^2 + (m/b)^2}}$$

22. 30

Are these functions equal?! No way!

one is single sum, other is double sum!

It is a tedious exercise to show that they are the same.

Basically need to show

~~y' not present~~

$$\frac{2}{a}\frac{\sinh\frac{n\pi(b-y+y')}{a}}{\sinh n\pi b/a} = \frac{16}{\pi a b}\sum_m \frac{\sin m\pi y/b \ \sin m\pi y'/b}{(n/a)^2+(m/b)^2}$$

ie work out Fourier expansion of function on left

and get coefficients ie must show

$$\underbrace{\frac{2}{b}\frac{2}{a}\int_{y'}^{y'+b}\frac{\sinh n\pi(b-y+y')/a}{\sinh n\pi b/a}\,dy\,\sin\frac{m\pi y}{b}} = \frac{16}{\pi a b}\frac{\sin m\pi y'/b}{(n/a)^2+(m/b)^2}$$

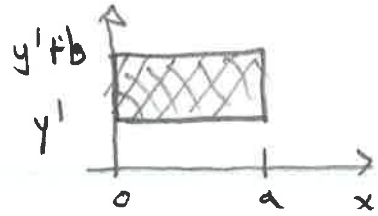
$$= \frac{4}{\pi}[\ \ ]$$

$$\int_{y'}^{y'+b}\sinh\frac{n\pi(b-y+y')}{a}\sin\frac{m\pi y}{b}\,dy$$

$$= \int_{y'}^{y'+b}\frac{1}{2}\frac{1}{2i}dy\left[e^{n\pi(b-y+y')/a}-e^{-n\pi(b-y+y')/a}\right]\left[e^{im\pi y/b}-e^{-im\pi y/b}\right]$$

$$= \frac{1}{4i}\left[\frac{e^{n\pi(b-y+y')/a}\,e^{im\pi y/b}}{-n\pi/a+im\pi/b} - \frac{e^{-n\pi(b-y+y')/a}\,e^{im\pi y/b}}{+n\pi/a+im\pi/b}\right]_{y'}^{y'+b}$$

$$= \frac{1}{4i}\left[\frac{e^{im\pi}e^{imy'/b}}{-\frac{n\pi}{a}+\frac{im\pi}{b}} - \frac{e^{-n\pi b/a}\,e^{imy'/b}}{-n\pi/a+im\pi/b}\ \cdots\right.$$



Moral (Again)

Laplace in $d=2$

G for $\nabla^2\psi = 0$ $\quad \psi(x,y) = \psi_s(x) = \psi(x, y=y')$

$$\psi(x,y) = \int dx'\, \psi_s(x')\, G(x-x', y-y')$$

G for $\nabla^2\psi = -4\pi\rho(x,y)$ identical

$$\psi(x,y) = \int dx'dy'\, \rho(x'y')\, G(x-x', y-y')$$

Diffusion Eqn $d=1$ $\quad D\dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial f}{\partial t} = 0$ $\quad f_s(x) = f(x, t=0)$

$$f(x,t) = \int_{-\infty}^{\infty} dx'\, f_s(x')\, G(x-x', t)$$

$$\underbrace{\frac{1}{2\pi}\sqrt{4\pi/Dt}}_{\frac{1}{\sqrt{4\pi Dt}}}\, e^{-(x-x')/4Dt}$$

$f_s(x') = \delta(x')$ $\quad f(x,t) = \dfrac{1}{\sqrt{4\pi Dt}}\, e^{-x^2/4Dt}$

$$\int_{-\infty}^{\infty} f(x,t)\,dx = \frac{1}{\sqrt{4\pi Dt}}\sqrt{4\pi Dt} = 1 \quad \checkmark$$

$$D\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = \rho(x,t)$$ ← Source of heat at x, t.

Claim: $\quad f(x,t) = \displaystyle\int_{-\infty}^{\infty} dx' \int_{-\infty}^{t} dt'\, \rho(x't')\, G(x, x', t, t')$

A bit different from Laplace in symmetry $f(x,-t)$ not a soln of Diffusion eqn if $f(x,t)$ is.

where $g(x\,x'\,t\,t') = \begin{cases} e^{-(x-x')^2/4D(t-t')} \dfrac{1}{\sqrt{4\pi D(t-t')}} & t'<t \\ 0 & t'>t \end{cases}$

if $D\dfrac{\partial^2 g}{\partial x^2} - \dfrac{\partial g}{\partial t} = \delta(x-x')\,\delta(t-t')$

We would be done since

$$D\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = \int dx' \int dt'\, \rho(x',t')\,\delta(x-x')\,\delta(t-t') = \rho(x\,t)$$

How to show this? Fourier

$$* \quad \int_{-\infty}^{\infty} \frac{e^{ikx}\,dk}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}\,d\omega}{2\pi}\, A(k,\omega) = g(x;t) = \frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}}$$

$$\int_{-\infty}^{\infty} \frac{e^{ikx}\,dk}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}\,d\omega}{2\pi}\, 1 = \delta(x)\,\delta(t)$$

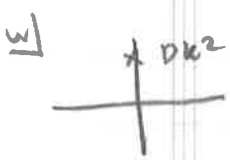
Apply $D\dfrac{\partial^2}{\partial x^2} - \partial/\partial t$ to $*$

$$\int_{-\infty}^{\infty} \frac{e^{ikx}\,dk}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}\,d\omega}{2\pi}\,(-Dk^2 - i\omega)\,A(k,\omega) =$$

$$(-Dk^2 - i\omega)\,A(k,\omega) = 1$$

$$A(k,\omega) = \frac{-1}{Dk^2 + i\omega}$$

$$\int_{-\infty}^{\infty} \frac{e^{ikx}\,dk}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}\,d\omega}{2\pi}\, \underbrace{\frac{-1}{Dk^2+i\omega}}_{-\frac{1}{i}\left(\frac{1}{\omega - iDk^2}\right)} = \int_{-\infty}^{\infty} \frac{e^{ikx}}{2\pi}\,\frac{1}{2\pi}\,2\pi i\left(\frac{-1}{i}\right) e^{-Dk^2 t}$$



Did this integral before . . .

$d=1$

Did all this in $d=2$, but much of E/M is in $d=3$.

Let's say a little about that...