Green's functions

We will begin in $d=2$ where we can draw nicer pictures + build intuition.

Solving a differential equation or partial differential equation involves not only specifying the eqn but also the boundary conditions.

$$m\frac{d^2x}{dt^2} = -mg \quad + \quad x(t=0) \quad \dot{x}(t=0)$$

$$\Longrightarrow \quad x(t) = x_0 + v_0 t - \tfrac{1}{2} g t^2$$

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$$

$$f(x,t) = A(x)\,B(t)$$

$$A\frac{dB}{dt} = DB\cdot\frac{d^2A}{dx^2} \qquad \frac{1}{D}\frac{1}{B}\frac{dB}{dt} = \frac{1}{A}\frac{d^2A}{dx^2} = -k^2$$

The sign was incorporating some intuition about boundary conditions

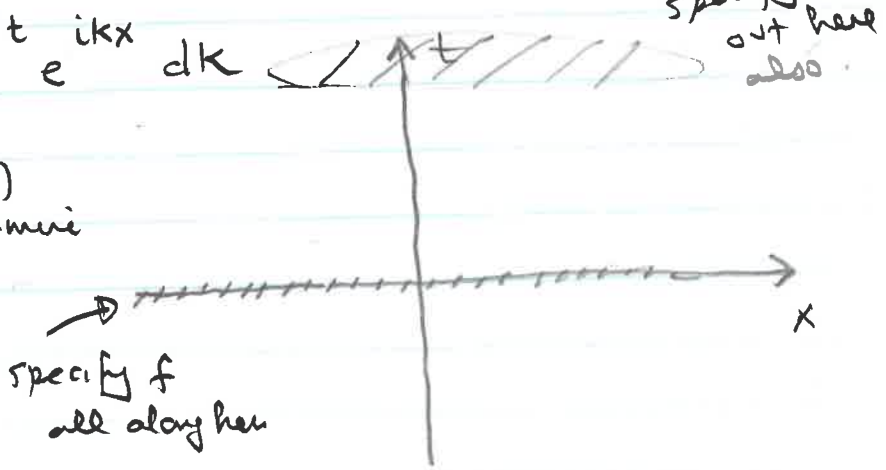
$$\frac{dB}{dt} = -Dk^2 B \qquad B(t) = e^{-Dk^2 t}$$

We could have chosen $+k^2$ in principle, but soln's would grow exponentially in time

Our soln also involved a knowledge of $f(x,0)$

$$f(x,t) = \int c(k)\, e^{-Dk^2 t}\, e^{ikx}\, dk$$

need $f(x,0)$ to determine



Consider wave eqn
for vibrating string

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$$

$$y = A(x)\,B(t)$$

$$B\frac{d^2A}{dx^2} = \frac{1}{v^2} A \frac{d^2B}{dt^2}$$

$$\frac{1}{A}\frac{d^2A}{dx^2} = \frac{1}{v^2}\frac{1}{B}\frac{d^2B}{dt^2} = -k^2$$

again physics

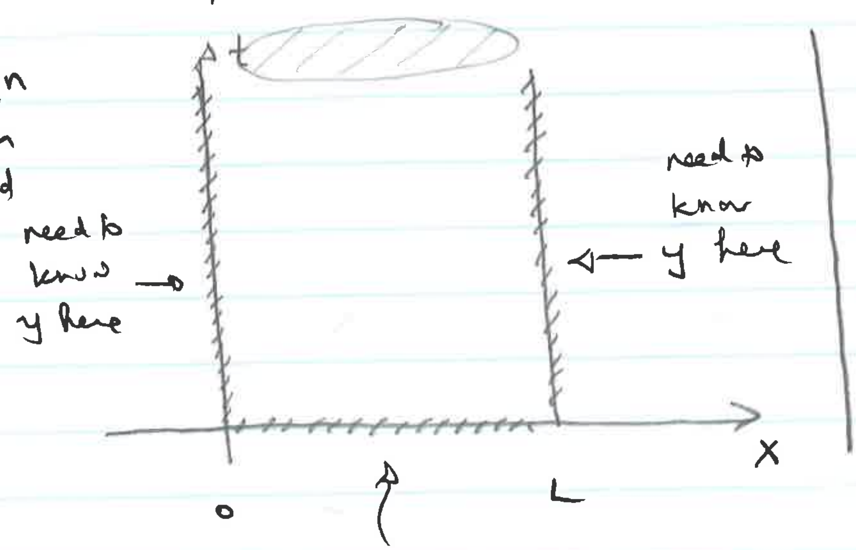
$A =$ $\sin kx$ $\cos kx$ $\qquad B = \sin kvt$ $\cos kvt$

more boundary conditions : $A(0) = 0$ $\quad A(L) = 0$

discard $\cos kx$ solns
and restrict $k \Rightarrow k_n = \frac{\pi}{L} n$

$$y(x,t) = \sum_n \sin\frac{\pi x}{L} n \left(a_n \cos\frac{\pi t n v}{L} + b_n \sin\frac{\pi t n v}{L}\right)$$

Wave eqn
needs open
"u" shaped
bdy



need to know y and $\frac{\partial y}{\partial t}$ here

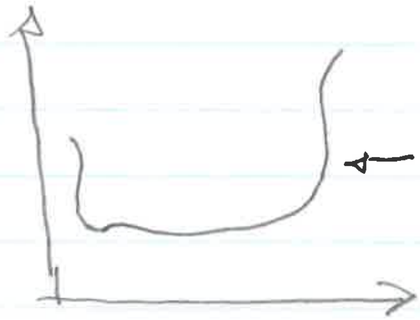
$$y(x,0) = \sum_n a_n \sin\frac{n\pi x}{L} \qquad a_m = \frac{2}{L}\int_0^L \sin\frac{m\pi x'}{L}\, y(x',0)\,dx'$$

$$y(x,t) = \sum_n \frac{2}{L}\int_0^L \sin\frac{n\pi x'}{L}\, y(x',0)\,dx' \,\sin\frac{n\pi x}{L}\cos\frac{n t n v}{L}$$

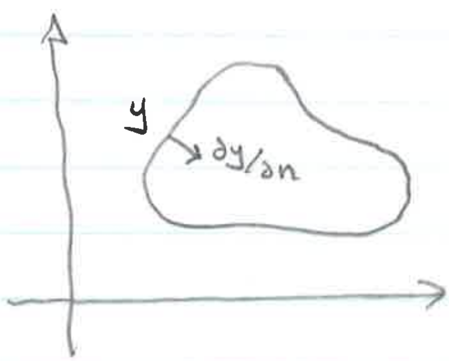
"Cauchy bdy conditions" : Specify value of function and its normal derivative ($\partial y / \partial t$ is change of y ⊥ to boundary at $t = 0$)

Questions:

(1) Can you use more bizarre boundaries



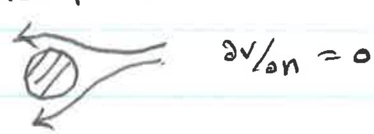
(2) What if curve is closed
would specifying y and $\partial y / \partial n$ everywhere be overly restrictive?



Ans: yes! only need y or $\partial y / \partial n$ if bdy closed

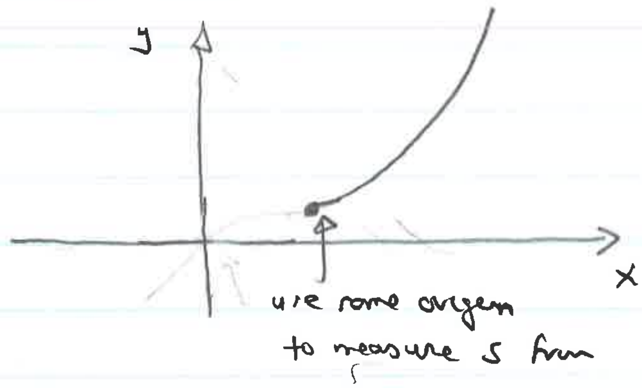
y : Dirichlet ← Laplace eqn with Y specified

$\partial y / \partial n$: Neumann ← fluid flow around solid bodies

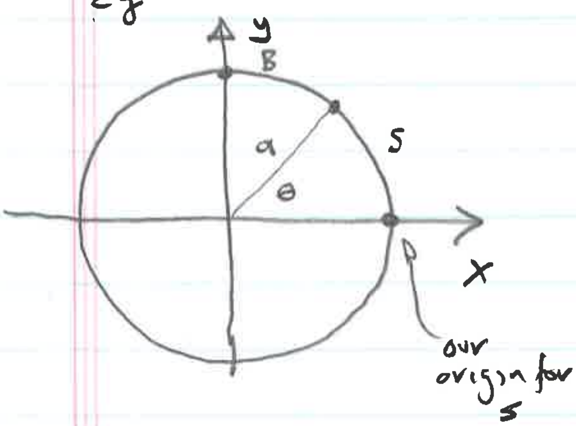


How to express a general boundary?

$$X = \xi(s) \quad Y = \eta(s)$$



Eg



$$x = a\cos\theta = a\cos(s/a) \equiv \xi(s)$$

$$y = a\sin\theta = a\sin(s/a) \equiv \eta(s)$$

Eg at "B" $s = \frac{2\pi}{4}a = \frac{\pi}{2}a, \theta = \frac{\pi}{2}$

Let's try computing $\psi(x,y)$ some distance from the body at (ξ,η)

$$\psi(x,y) = \psi(\xi,\eta) + (x-\xi)\frac{\partial\psi}{\partial x} + (y-\eta)\frac{\partial\psi}{\partial y}$$

$$+ \frac{1}{2}\left[(x-\xi)^2\frac{\partial^2\psi}{\partial x^2} + 2(x-\xi)(y-\eta)\frac{\partial^2\psi}{\partial x\partial y} + (y-\eta)^2\frac{\partial^2\psi}{\partial y^2}\right]$$

If we knew <u>all</u> the partial derivatives then we would know ψ

We shall not need <u>all</u> of them, as our examples have already hinted.

What do we know?

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simple analogy   Is this analogous?

$$m\frac{d^2x}{dt^2} = -mg$$

$$x(t) = x(0) + \left.\frac{dx}{dt}\right|_0 t + \frac{1}{2}\left.\frac{d^2x}{dt^2}\right|_0 t^2 + \frac{1}{6}\left.\frac{d^3x}{dt^3}\right|_0 t^3 + \dots$$

Need to specify $x(0), \left.\frac{dx}{dt}\right|_0, \left.\frac{d^2x}{dt^2}\right|_0, \dots$!?

why do we not need higher ones?

The partial differential eqn gives the $\left.\frac{d^2x}{dt^2}\right|_{t=0} = -g$

all higher ones zero!

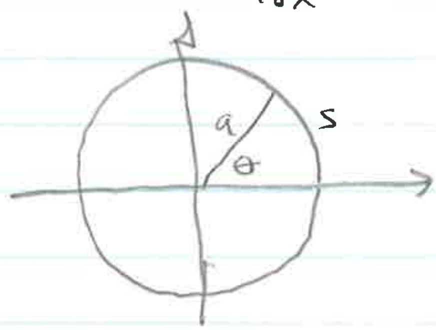
Something similar here

If we are told ψ along the curve $[\xi(s), \eta(s)]$ then we can always compute $\psi(s) = \psi[\xi(s), \eta(s)]$

$$\frac{d\psi}{ds} = \frac{\partial\psi}{\partial x}\frac{d\xi}{ds} + \frac{\partial\psi}{\partial y}\frac{d\eta}{ds}$$ ← one eqn relating $\partial\psi/\partial x$ and $\partial\psi/\partial y$

Do Later @ *

Example! $\psi = 0$ on circle



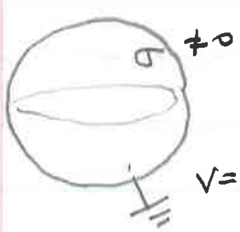
$$\frac{d\xi}{ds} = -\sin s/a$$

$$d\eta/ds = +\cos s/a$$

$$-\partial\psi/\partial x \sin s/a + \partial\psi/\partial y \cos s/a = 0$$

~~one eqn relating~~

3-d analogy



$E \sim \sigma$

$\uparrow$

$\vec{\nabla} V$

suppose also we are told normal derivative of ψ is a nonzero constant σ

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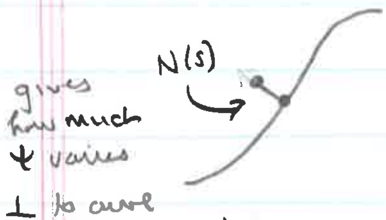
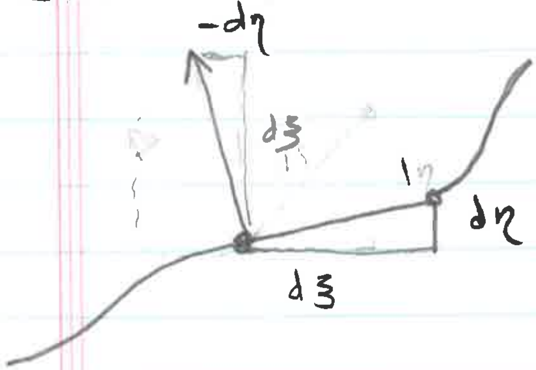
If we are also told how ψ changes $\perp$ to curve

Denote $N(s)$ as rate of change of ψ in direction $\perp$ to curve



$\xi(s), \eta(s)$

$$N(s) = \left(\frac{\partial\psi}{\partial x}\frac{d\eta}{ds} - \frac{\partial\psi}{\partial y}\frac{d\xi}{ds} \right)$$



normal is $-d\eta$

solve these 2 eqns for $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial y}$!

$$* \quad +\frac{\partial\psi}{\partial x}\cos s/a + \frac{\partial\psi}{\partial y}\sin s/a = \sigma \qquad \cos \qquad \sin$$

$$-\partial\psi/\partial x \sin s/a + \partial\psi/\partial y \cos s/a = 0 \qquad -\sin \qquad +\cos$$

$$\partial\psi/\partial x = \sigma\cos s/a$$
$$\partial\psi/\partial y = \sigma\sin s/a$$

$$\frac{d\psi}{ds} = \frac{\partial\psi}{\partial x}\frac{d\xi}{ds} + \frac{\partial\psi}{\partial y}\frac{d\eta}{ds} \qquad d\xi/ds \qquad d\eta/ds$$

$$N(s) = \frac{\partial\psi}{\partial x}\frac{d\eta}{ds} - \frac{\partial\psi}{\partial y}\frac{d\xi}{ds} \qquad +d\eta/ds \qquad -d\xi/ds$$

$$\left.\begin{aligned} \frac{\partial\psi}{\partial x} &= N(s)\frac{d\eta}{ds} + \frac{d\xi}{ds}\frac{d\psi}{ds} \equiv p(s) \\ \frac{\partial\psi}{\partial y} &= -N(s)\frac{d\xi}{ds} + \frac{d\eta}{ds}\frac{d\psi}{ds} \equiv q(s) \end{aligned}\right\}$$

NB $(d\xi/ds)^2 + (d\eta/ds)^2 = 1$!

known functions since $N(s)$, $\frac{d\psi}{ds}$, η, ξ all known

7

Getting 2nd derivatives is crucial. If we get 2nd derivs, higher ones just "more of the same"

Three "unknowns" $\frac{\partial^2\psi}{\partial x^2}$; $\frac{\partial^2\psi}{\partial x\partial y}$; $\frac{\partial^2\psi}{\partial y^2}$

2 Eqns immediately

$x = \xi(s)$

Remember $p(s)$ is known!

$$\frac{dp}{ds} = \frac{d}{ds}\frac{\partial\psi}{\partial x} = \frac{\partial^2\psi}{\partial x^2}\frac{d\xi}{ds} + \frac{\partial^2\psi}{\partial x\partial y}\frac{d\eta}{ds}$$

$$\frac{dq}{ds} = \quad = \frac{\partial^2\psi}{\partial x\partial y}\frac{d\xi}{ds} + \frac{\partial^2\psi}{\partial y^2}\frac{d\eta}{ds}$$

$\times$ What is 3rd eqn?
What haven't we used at all yet?

$\boxed{\text{The PDE}}$

$$A(x,y)\frac{\partial^2\psi}{\partial x^2} + 2B(x,y)\frac{\partial^2\psi}{\partial x\partial y} + C(x,y)\frac{\partial^2\psi}{\partial y^2} = F$$

$$F\left(x, y, \psi, \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right)$$

If Eqn is linear $H(x,y) + D(x,y)\frac{\partial\psi}{\partial x}$

$+ E(x,y)\frac{\partial\psi}{\partial y} + G(x,y)\psi$

$$\begin{pmatrix} \frac{d\xi}{ds} & \frac{d\eta}{ds} & 0 \\ 0 & \frac{d\xi}{ds} & \frac{d\eta}{ds} \\ A & 2B & C \end{pmatrix}\begin{pmatrix} \frac{\partial^2\psi}{\partial x^2} \\ \partial^2\psi/\partial x\partial y \\ \frac{\partial^2\psi}{\partial y^2} \end{pmatrix} = \begin{pmatrix} dp/ds \\ dq/ds \\ F(s) \end{pmatrix}$$

Can show all higher derivatives can be computed in terms of known quantities.

* One caveat: Determinant $\neq 0$?

$$\frac{d\xi}{ds}\left[C\frac{d\xi}{ds} - 2B\,{}^{d\eta}/_{ds}\right] - \frac{d\eta}{ds}\left[-A\,{}^{d\eta}/_{ds}\right]$$

$$= C\left(\frac{d\xi}{ds}\right)^2 - 2B\frac{d\xi}{ds}\frac{d\eta}{ds} + A\left(\frac{d\eta}{ds}\right)^2$$

$$d\xi = dx \qquad d\eta = dy$$

$$C(x,y)\,dx^2 - 2B(x,y)\,dx\,dy + A(x,y)(dy)^2 = 0$$

Cauchy uniquely specifies soln unless the curve $\xi(s), \eta(s)$ is a "characteristic"! $\longrightarrow$ cont'd

It is interesting to understand why ~~[illegible]~~ giving ψ and N along a characteristic is insufficient.

One can discuss this problem generally, but maybe it is better to take the example of the wave eqn

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \qquad \begin{aligned} A(x,t) &= 1 \\ B(x,t) &= 0 \\ C(x,t) &= -1/c^2 \end{aligned}$$

$$-1/c^2\,dx^2 + dt^2 = 0$$

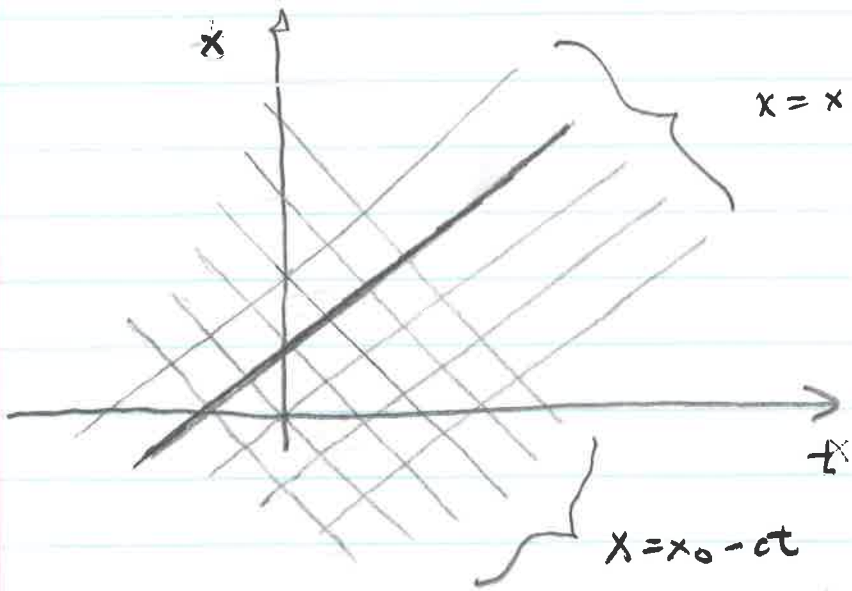
$$\left(\frac{dx}{dt}\right)^2 = c^2 \qquad \frac{dx}{dt} = \pm c \qquad x = x_0 \pm ct$$

The solution to the wave eqn is the sum :

$$\psi(x,t) = f(x+ct) + g(x-ct)$$

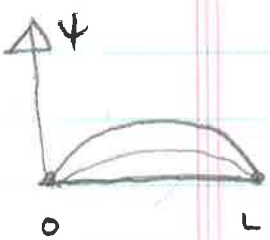
arbitrary functions

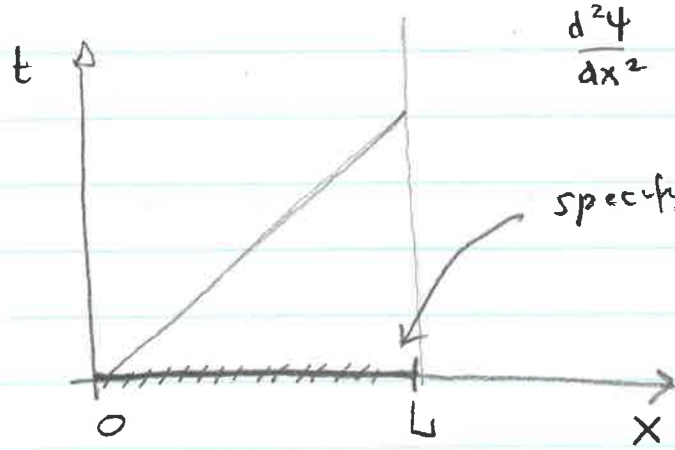
Giving ψ along $x = x_0 \pm ct$ and $N(s)$ along these will only specify one f or g and not the other



One needs a bdy which cuts across

More complex pdes may have / will have more complicated characteristics.





$$\frac{d^2\psi}{dx^2} - \frac{1}{c^2}\frac{d^2\psi}{dt^2} = 0$$

specifying ψ and $\frac{d\psi}{dt}$ here is good

Actually, there are always two families of characteristics:

$$A\,dy^2 - 2B\,dx\,dy + C\,dx^2 = 0$$

$$dy = \left[2B\,dx \pm \sqrt{4B^2dx^2 - 4AC\,dx^2}\right] / 2A$$

$$A\,dy = \left[B \pm \sqrt{B^2 - AC}\right] dx$$

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

wave eqn $\quad A = 1 \quad C = -1/c^2 \quad B = 0$

$$\frac{dt}{dx} = \frac{0 \pm \sqrt{1/c^2}}{1} = \pm 1/c$$

(10)

very explicitly:

(11)

$$\psi(x,t) = \sum_n \sin\frac{n\pi x}{L}\left(a_n \cos\frac{n\pi v t}{L} + b_n \sin\frac{n\pi v t}{L}\right)$$

$$= \sum_n \frac{a_n}{2}\left\{\sin\frac{n\pi}{L}(x+vt) - \sin\frac{n\pi}{L}(x-vt)\right\}$$

$2\sin A\cos B$
$= \sin(A+B) - \sin(A-B)$

$$+ \frac{b_n}{2}\left\{\cos\frac{n\pi}{L}(x-vt) - \cos\frac{n\pi}{L}(x+vt)\right\}$$

$2\sin A\sin B$
$= +\cos(A-B) - \cos(A+B)$

$$= \sum_n \left[\frac{a_n}{2}\sin\frac{n\pi}{L}(x+vt) - \frac{b_n}{2}\cos\frac{n\pi}{L}(x+vt)\right] \leftarrow f(x+vt)$$

$$+ \left[-\frac{a_n}{2}\sin\frac{n\pi}{L}(x-vt) + \frac{b_n}{2}\cos\frac{n\pi}{L}(x-vt)\right] \leftarrow g(x-vt)$$

given $\psi(x,t)$ along $x = x_0 + vt$ only $x - vt = x_0 \ldots$

?

~~Boston Tire~~

characteristics also taught us that $B^2 - AC$ is an impt quantity

$B(x,y)^2 > A(x,y)\,C(x,y)$ everywhere : hyperbolic

$0 > 1\,(-1/c^2)$ wave eqn

$AC > B^2$ elliptic

(everywhere)

$AC = B^2$ parabolic

Diffusion

$\frac{\partial \psi}{\partial t} = D \frac{\partial^2 \psi}{\partial x^2}$

Laplace

$\frac{d^2 \psi}{dx^2} + \frac{d^2 \psi}{dy^2}$

$A = 1 \quad C = 1 \quad B = 0$

Bottom Line

Conditions	Bdy	Hyperbolic	Elliptic	Parabolic
Dirichlet or Neumann (value or slope)	OPEN	insufficient	insufficient	~~unique~~ sufficient
	CLOSED	sufficient	sufficient	overspecified
CAUCHY value and slope	OPEN	sufficient	sufficient	overspecified
	CLOSED	overspecified	overspecified	overspecified

Now finally Green's functions ...

We will consider $d=2$ Laplace Eqn

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

Elliptic: we claimed Dirichlet or Neumann on closed surface sufficient

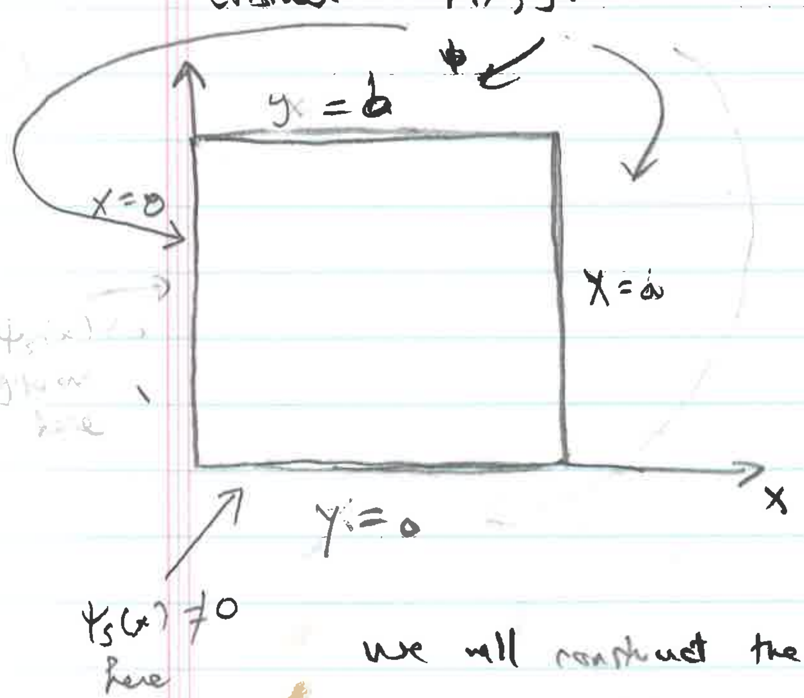
$$\psi = X(x)\,Y(y)$$

$$\underbrace{\frac{X''}{X}}_{} = -k^2 = -\frac{Y''}{Y}$$

$\sin kx$ $\qquad\qquad$ $\sinh ky$

$\cos kx$ $\qquad\qquad$ $\cosh ky$

Consider $\psi(x,y)=0$ along $x=0$ $\quad x=a$ $\quad y=b$



We will construct the Green function for this PDE and these BC.

Emphasize $\quad \text{PDE} \rightarrow G \leftarrow \text{BC}$

can guarantee b.c $x=0, x=a$ by choosing $k = \frac{\pi n}{a}$ and

taking only the sinlex soln

$$\psi(x,y) = \sum b_n \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}(b-y)$$

Also can guarantee $y=b$ bc by choosing $\sinh(b-y)k$

$$\Big[\sinh(A+B) = \tfrac{1}{2}\left[e^{A+B} - e^{-A-B}\right]$$
$$= \tfrac{1}{4}(e^A - e^{-A})(e^B + e^{-B}) + \tfrac{1}{4}(e^A + e^{-A})(e^B - e^{-B})$$
$$= \sinh A \cosh B + \cosh A \sinh B \Big]$$

$$\psi_s(x) \equiv \psi(x, y=0) = \sum b_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}$$

$$\int_0^a dx \sin\frac{m\pi x}{a}\psi_s(x) = \frac{a}{2} b_n \sinh\frac{n\pi b}{a}$$

$$\int_0^a \sin^2\frac{n\pi x}{a}dx$$
$$\Rightarrow \theta = \frac{n\pi x}{a}$$
$$\int_0^{n\pi} \sin^2\theta \, d\theta \, \frac{a}{n\pi}$$
$$= \frac{n\pi}{2}\frac{a}{n\pi} = \frac{a}{2}$$

$$\boxed{\psi(x,y) = \sum_{n=1}^{\infty}\left(\frac{2}{a}\int_0^a dx' \frac{\sin n\pi x'}{a}\psi_s(x')\right)\sin\frac{\pi n x}{a} \frac{\sinh {}^{n\pi}/_a (b-y)}{\sinh \pi n b/a}}$$

~~Consider limits $b \to \infty$ and $a \to \infty$ (soln of Laplace eqn~~
~~in uhp given ψ along $x=0$)~~

$$\psi(x,y) = \int dx' \psi_s(x') \underbrace{G(x-x', y)}_{\frac{2}{a}\sum_{n=1}^{\infty}\frac{\sinh {}^{n\pi}/_a(b-y)}{\sinh {}^{n\pi}/_a b}\left(\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\right)}$$