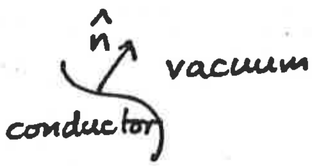
FOR A CONDUCTING SURFACE, IT MAY BE THAT THE TOTAL CHARGE IS SPECIFIED, NOT THE POTENTIAL φ_0 OF THE SURFACE. IN THIS CASE, SINCE THE SURFACE IS AN EQUIPOTENTIAL, TAKE THE POTENTIAL TO BE φ_0, SOLVE FOR $\varphi(\vec{r})$, THEN DETERMINE THE SURFACE CHARGE DENSITY $\sigma(\vec{r}) = -\frac{1}{4\pi}\,\hat{n}\cdot\vec{\nabla}\varphi(\vec{r}) = \frac{E_N}{4\pi}$



SO THAT THE TOTAL CHARGE IS

$= \frac{E_N}{4\pi}$

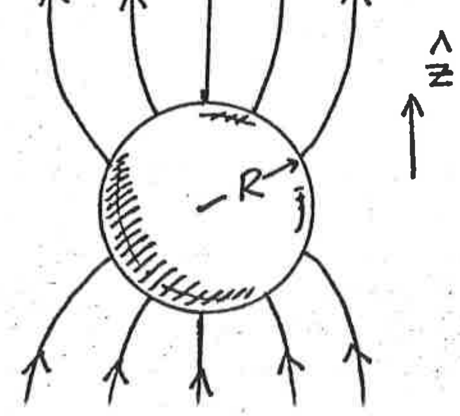
$$Q = \int_S dS\,\sigma(\vec{r})$$

$$= -\frac{1}{4\pi}\int_S dS\,\hat{n}\cdot\vec{\nabla}\varphi(\vec{r}).$$

THEN ADJUST φ_0 TO MAKE Q EQUAL TO THE SPECIFIED CHARGE.

LET'S NOW CONSIDER A THIRD METHOD OF SOLVING THE LAPLACE EQUATION $\nabla^2\varphi = 0$, THE METHOD OF *separation of variables*. THIS METHOD IS ONE YOU'VE SEEN BEFORE BUT IS WORTH REVIEWING. THE IDEA IS TO WRITE THE SOLUTION FOR $\varphi(\vec{r})$ AS A PRODUCT OF THREE FUNCTIONS, each depending only on a single coordinate, ONE FOR EACH DIMENSION OF SPACE. EACH OF THE THREE FUNCTIONS SATISFIES AN ORDINARY DIFFERENTIAL EQUATION, SO THAT WE'VE REPLACED OUR SINGLE PARTIAL DIFFERENTIAL EQUATION BY THREE ORDINARY DIFFERENTIAL EQUATIONS... A TRADEOFF THAT HOPEFULLY (AND USUALLY) MAKES LIFE EASIER.

LET'S ILLUSTRATE BY CONSIDERING A SPECIFIC EXAMPLE.



AN UNCHARGED CONDUCTING SPHERE IS PLACED IN A REGION OF SPACE WHERE INITIALLY THERE WAS A UNIFORM ELECTRIC FIELD $\vec{E}(\vec{r}) = E\hat{z}$. WHAT IS THE POTENTIAL EVERYWHERE IN SPACE IN THE PRESENCE OF THE SPHERE?

WE FIRST NOTE THAT THE PROBLEM IS AZIMUTHALLY SYMMETRIC, SO THAT $\varphi = \varphi(r,\theta)$ ONLY, AND SO TO FIND $\varphi(\vec{r})$ IN THE REGION OUTSIDE THE SPHERE WE NEED SOLVE LAPLACE'S EQUA IN ONLY 2 DIMENSIONS:

$$\nabla^2 \varphi(r,\theta) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\varphi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \, \frac{\partial\varphi}{\partial\theta}\right) = 0$$

IN THE REGION $r > R$ (outside the sphere)

SINCE THE SPHERE IS CONDUCTING, ONCE WE KNOW φ ON THE SURFACE, WE KNOW φ EVERYWHERE INSIDE THE SPHERE. THE BOUNDARY CONDITIONS ON THE DIFFERENTIAL EQUATION ARE

(i) AT $r = R$, φ IS INDEPENDENT OF θ; AND

(ii) AS $r \to \infty$, φ GOES OVER TO THE FORM APPROPRIATE TO $\vec{E}(\vec{r}) = E\hat{z}$, i.e.

$$\varphi \to -Ez = -Er\cos\theta \quad \text{AS} \quad r \to \infty.$$

TO PROCEED, WE TRY WRITING $\varphi(r,\theta)$ AS A PRODUCT: $R(r)\,\Theta(\theta)$. DOES THIS WORK? WE GET

$$\frac{\Theta}{r}\frac{d^2}{dr^2}(rR) + \frac{R}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) = 0,$$

OR, MULTIPLYING BY $r^2/R\Theta$,

$$\underbrace{\frac{r}{R}\frac{d^2}{dr^2}(rR)}_{\text{function of } r \text{ alone}} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)}_{\text{function of } \theta \text{ alone}} = 0.$$

FOR THIS TO WORK, EACH PIECE MUST IN FACT BE A *number* INDEPENDENT OF r OR θ. WE LET THE FUNCTION OF r BE EQUAL TO $\ell(\ell+1)$, WHERE ℓ IS AS YET UNDETERMINED

THEN WE HAVE TWO ORDINARY DIFFERENTIAL EQUATIONS:

$$\frac{d^2}{dr^2}(rR) = \frac{\ell(\ell+1)R}{r},$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\Theta = 0.$$

THE θ EQUATION IS JUST LEGENDRE'S EQUATION, WITH SOLUTION

$$\Theta(\theta) = CP_\ell(\cos\theta) + DQ_\ell(\cos\theta).$$

HERE $P_\ell(x)$ AND $Q_\ell(x)$ ARE THE *Legendre functions of the first and second kind*. NOW, WE WANT $\varphi(\vec{r})$ AND THUS $\Theta(\theta)$ TO BE WELL-BEHAVED EVERYWHERE: HOWEVER, THE $Q_\ell(x)$ *diverge* FOR $x=\pm1$ *(corresponding to* $\theta=0,\pi$*)*, SO WE MUST SET $D=0$. FOR THE SOLUTIONS OF THE FIRST KIND, WE FIND THAT THE P_ℓ *also* DIVERGE FOR $x=\pm1$ *unless* $\ell=0,1,2,\ldots$ WITH THIS CHOICE OF ℓ, THE P_ℓ ARE JUST THE USUAL LEGENDRE *polynomials*:

$$P_0(x)=1,$$
$$P_1(x)=x,$$
$$P_2(x)=\tfrac{1}{2}(3x^2-1),$$
etc.

THUS *physically* WE SEE THAT WE MUST HAVE ℓ BE A NONNEGATIVE INTEGER: THIS EXPLAINS OUR CURIOUS CHOICE OF $\ell(\ell+1)$ AS THE CONSTANT WE USED IN SEPARATING VARIABLES.

FOR ℓ INTEGER, THE SOLUTION TO THE RADIAL EQUATION IS

$$R(r) = Ar^\ell + Br^{-(\ell+1)}.$$

THE QUESTION THAT THEN ARISES IS: *what value of ℓ do we choose?* THE ANSWER IS SIMPLE: *we don't know*. TO MAKE SURE THAT OUR EXPRESSION FOR $\varphi(\vec{r})$ IS ABLE TO FIT THE DESIRED BOUNDARY CONDITIONS, WE MAKE USE OF THE SUPER-

POSITION PRINCIPLE AND WRITE $\varphi(\vec{r})$ AS A SUM OF THE SOLUTIONS FOR *all* ℓ:

$$\varphi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell(\cos\theta)$$

THE COEFFICIENTS A_ℓ AND B_ℓ REMAIN UNDETERMINED: THEY'LL BE CHOSEN TO FIT THE BOUNDARY CONDITIONS. WE GAIN CONFIDENCE IN OUR GENERAL SOLUTION OF $\nabla^2\varphi = 0$ FOR AZIMUTHALLY SYMMETRIC PROBLEMS BY NOTING THE FOLLOWING: THE INFORMATION IN THE TERM FOR A GIVEN ℓ IS <u>not</u> CONTAINED IN THE TERM FOR ANY OTHER ℓ.* THIS IS BECAUSE THE P_ℓ ARE *orthogonal*. RECALL THAT FOR ORDINARY UNIT VECTORS, ORTHOGONALITY REFERS TO SUMMING OVER COMPONENTS:

$$\sum_{i=1}^{3} (\hat{e}_j)_i (\hat{e}_k)_i = 0 \quad \text{FOR } j \neq k$$

THE "COMPONENTS" OF THE $P_\ell(x)$ ARE THE VALUES OF THE FUNCTION FOR EACH X: THE SUM THUS BECOMES AN INTEGRAL OVER ALL X FROM -1 TO 1.

$$\int_{-1}^{1} dx\, P_j(x)\, P_k(x) = 0 \quad \text{FOR } j \neq k.$$

or, in general, $\int_{-1}^{1} dx\, P_j(x) P_k(x) = \frac{2}{2j+1}\delta_{jk}$

JUST AS ORTHOGONALITY SAYS THAT THE X-COMPONENT OF A VECTOR TELLS YOU NOTHING ABOUT THE Y-COMPONENT, SO KNOWING THE A_ℓ AND B_ℓ FOR A GIVEN ℓ TELLS YOU NOTHING ABOUT THE COEFFICIENTS FOR OTHER ℓ. NOTE THAT THE P_ℓ ARE ORTHOGONAL BUT NOT ORTHONORMAL:

$$\int_{-1}^{1} d\cancel{x}\, P_\ell(x)\, P_\ell(x) = \frac{2}{2\ell+1} \neq 1.$$

THE $P_\ell(x)$ ARE ALSO *complete*: <u>any</u> FUNCTION IN $-1 \leq x \leq 1$ CAN BE EXPRESSED AS A SUM OF $P_\ell(x)$. SO OUR GENERAL,

AZIMUTHALLY-SYMMETRIC SOLUTION SEEMS IN FACT TO BE QUITE GENERAL.

LET US NOW RETURN TO THE PROBLEM OF THE SPHERE: FINDING $\varphi(\vec{r})$ FOR $r \geq R$ HAS BEEN REDUCED TO THE PROBLEM OF FINDING THE A_ℓ AND B_ℓ (there's still an infinite set of unknowns). WE DETERMINE THESE FROM THE BOUNDARY CONDITIONS:

(i) $\varphi(r,\theta)$ IS INDEPENDENT OF θ FOR $r=R$.

SINCE $P_0(\cos\theta)=1$ AND ALL OTHER $P_\ell(\cos\theta)$ DEPEND ON θ, WE HAVE

$$A_\ell R^\ell + B_\ell R^{-(\ell+1)} = 0 \quad \text{FOR ALL } \ell \geq 1, \text{ OR}$$

$$B_\ell = -A_\ell R^{2\ell+1} \quad \text{FOR } \ell \geq 1.$$

(ii) AS $r \to \infty$, $\varphi(r,\theta) \to -Er\cos\theta$.

AS $r \to \infty$ ONLY THE $A_\ell r^\ell P_\ell(\cos\theta)$ TERMS SURVIVE: SINCE $P_1(\cos\theta) = \cos\theta$, WE SEE THAT

$$A_1 = -E, \qquad \text{ALL OTHER } A_\ell = 0.$$

THUS $$B_1 = ER^3, \qquad \text{ALL OTHER } B_\ell = 0,$$

AND SO

$$\varphi(r,\theta) = -Er\cos\theta + \frac{ER^3}{r^2}\cos\theta$$

external POTENTIAL THAT WAS PRESENT BEFORE THE SPHERE WAS INTRODUCED

That this vanishes as $1/r^2$ tells us immediately no net charge on conductor

induced POTENTIAL DUE TO CHARGE DISTRIBUTION INDUCED ON THE SURFACE OF THE SPHERE

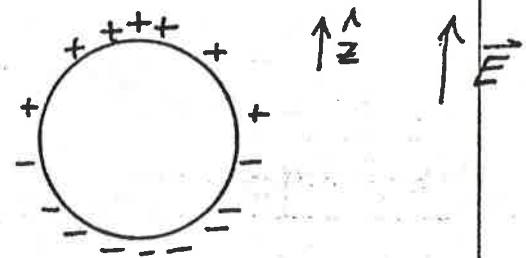
NOTE THAT AT $r=R$, THE POTENTIAL IS $\varphi=0$: SINCE THE SPHERE IS CONDUCTING AND THUS AN EQUIPOTENTIAL, $\varphi=0$ FOR ALL $r \leq R$.

WE CAN FINALLY DETERMINE THE INDUCED SURFACE CHARGE DENSITY ON THE SPHERE: THIS IS

$$\sigma(\theta) = \frac{1}{4\pi} E_r \Big|_{r=R} = -\frac{1}{4\pi} \frac{\partial \varphi}{\partial r}\Big|_{r=R}$$

$$= \underbrace{+\frac{E}{4\pi}\cos\theta}_{\text{from external term}} + \underbrace{\frac{E}{2\pi}\cos\theta}_{\text{from induced term}}$$

$$= \frac{3E}{4\pi}\cos\theta$$



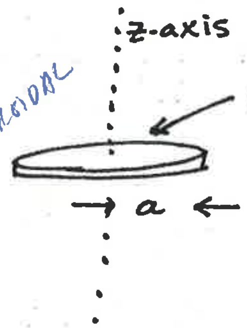
SO THE + CHARGES ARE ON THE +Z SIDE AND THE − CHARGES ON THE −Z SIDE, WHICH MAKES SENSE.

THE TOTAL CHARGE ON THE SPHERE IS

$$\int_S dS\, \sigma = 0,$$ WHICH AGREES WITH OUR ORIGINAL DESCRIPTION OF THE SPHERE AS uncharged.

AS A SECOND APPLICATION, LET'S RETURN TO THE PROBLEM OF THE FIELD DUE TO A CHARGED DISC OF RADIUS a: ON THE z (symmetry) AXIS, WE FOUND (p. 16) THAT

SEE LANDAU LIFSHITZ (OBLATE-SPHEROIDAL COORDINATE)



$$\varphi(0,0,z) = \varphi(\underset{=z}{r},\underset{=0}{\theta}) = 2\pi\sigma\left[\sqrt{a^2+z^2} - |z|\right]$$

THIS IS IN FACT ENOUGH TO NOW ALLOW US TO EXTEND THE SOLUTION TO ALL $\vec{r}$ WITH $r > a$... NOTE THAT ONLY FOR $r>a$ IS $\nabla^2\varphi = 0$ IN FACT SATISFIED. ON THE POSITIVE Z-AXIS, $\cos\theta = 1$ AND OUR GENERAL, AZIMUTHALLY-SYMMETRIC SOLUTION IS

$$\varphi(r,\theta=0) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell(1)$$

$$= \sum^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right)$$

SINCE $P_\ell(1)=1$ FOR ALL ℓ. ON THE POSITIVE z-AXIS, BY COMPARISON, THE SPECIAL SOLUTION FOUND ON P. 16 IS

$$\varphi(r,\theta=0) = 2\pi\sigma\left[\sqrt{a^2+r^2}-r\right]$$

$$= 2\pi\sigma r\left[\sqrt{1+a^2/r^2}-1\right] \qquad \frac{1}{z^{\ell+1}}$$

$$= \frac{\pi\sigma a^2}{r} + 2\pi\sigma \sum_{\substack{\ell=2\\ \text{even}}}^{\infty} (-1)^{\ell/2} \frac{(\ell-1)!!}{(\ell+2)!!} \frac{a^{\ell+2}}{r^{\ell+1}}$$

BY EXPANDING IN $a^2/r^2<1$ FOR $r>a$.

BY COMPARISON, WE SEE THAT

$A_\ell = 0$ FOR ALL ℓ ($\varphi\to 0$ AS $r\to\infty$),

$B_\ell = 0$ FOR ALL ODD ℓ (SYSTEM IS UP$\leftrightarrow$DOWN SYMMETRIC)

$B_0 = \pi\sigma a^2$ (total charge on disc of radius a),

$$B_\ell = 2\pi\sigma(-1)^{\ell/2}\frac{(\ell-1)!!}{(\ell+2)!!}a^{\ell+2}; \quad \ell \text{ EVEN}.$$

THUS WE KNOW $\varphi(r,\theta)$ FOR any $r>a$:

$$\varphi(r,\theta) = \frac{\pi\sigma a^2}{r} + 2\pi\sigma \sum_{\substack{\ell=2\\ \text{even}}}^{\infty} (-1)^{\ell/2} \frac{(\ell-1)!!}{(\ell+2)!!} \frac{a^{\ell+2}}{r^{\ell+1}} P_\ell(\cos\theta)$$

THUS KNOWING φ ALONG THE SYMMETRY AXIS IS ENOUGH TO TELL YOU φ FOR ANY θ. NOTE THAT $\varphi(r,\theta)$ AS ABOVE IS only VALID FOR $r>a$: FOR $r<a$ ONLY POSITIVE POWERS OF r CAN OCCUR, SINCE φ MUST REMAIN FINITE AS $r\to 0$. INDEED, FOR $r<a$ THE GENERAL, AZIMUTHALLY-SYMMETRIC SOLUTION IS NOT EVEN VALID, SINCE $\nabla^2\varphi=0$ IS NOT TRUE EVERYWHERE IN THIS REGION. TO GET $\varphi(\vec{r})$

EVERYWHERE IN SPACE, WE WOULD NEED A BETTER COORDINATE SYSTEM (THIS CAN BE DONE WITH OBLATE SPHEROIDAL COORDINATES — SEE LANDAU + LIFSHITZ, Electrodynamics of Continuous Media, SEC. 4).

THIS SAME PROCESS OF SEPARATION OF VARIABLES CAN BE CARRIED OUT FOR LAPLACE'S EQUATION FOR OTHER COORDINATE SYSTEMS (rectangular, cylindrical, etc.) AS WELL AS FOR SYSTEMS WITHOUT AZIMUTHAL SYMMETRY. FOR NONAZIMUTHALLY SYMMETRIC SYSTEMS, THE GENERAL SOLUTION FOR LAPLACE'S EQUATION $\nabla^2 \varphi = 0$ IN SPHERICAL COÖRDINATES IS

$$\varphi(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^{\ell} + B_{\ell m} r^{-(\ell+1)}\right) Y_{\ell m}(\theta,\phi),$$

WHERE THE $Y_{\ell m}(\theta,\phi)$ ARE CALLED spherical harmonics. THEY SATISFY THE DIFFERENTIAL EQUATION

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial Y_{\ell m}(\theta,\phi)}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{\ell m}(\theta,\phi)}{\partial\phi^2}$$

$$+ \ell(\ell+1) Y_{\ell m}(\theta,\phi) = 0$$

AND ARE JUST PRODUCTS OF A FUNCTION FOR θ AND A FUNCTION FOR ϕ:

$$Y_{\ell m}(\theta,\phi) = \underbrace{\sqrt{\frac{2\ell+1}{4\pi}\cdot\frac{(\ell-m)!}{(\ell+m)!}}}_{\text{factor so that } Y_{\ell m} \text{ are both orthogonal and normalized:}} P_\ell^m(\cos\theta)\, e^{im\phi}$$

m runs between $-\ell$ and ℓ

associated Legendre polynomial: reduces to P_ℓ for $m=0$

$$\int d\Omega\, Y^*_{\ell m}(\theta,\phi)\, Y_{\ell' m'}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}$$

THE $Y_{lm}(\theta,\phi)$ ARE complete IN THAT ANY FUNCTION OF θ AND ϕ MAY BE EXPRESSED IN TERMS OF A SUM OVER THE $Y_{lm}(\theta,\phi)$. THUS, FOR ANY FUNCTION $f(\theta,\phi)$,

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}\, Y_{lm}(\theta,\phi)$$

USING THE ORTHONORMAL PROPERTY, HOWEVER,

$$\int d\Omega\, f(\theta,\phi)\, Y^*_{l'm'}(\theta,\phi)$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm} \underbrace{\int d\Omega\, Y^*_{l'm'}(\theta,\phi)\, Y_{lm}(\theta,\phi)}_{\delta_{ll'}\delta_{mm'}}$$

$$= a_{l'm'},$$

SO THAT WE MAY WRITE OUR EXPANSION OF $f(\theta,\phi)$ AS

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left(\int_{-1}^{1} d(\cos\theta')\int_0^{2\pi} d\phi'\, f(\theta',\phi')\, Y^*_{lm}(\theta',\phi')\right) Y_{lm}(\theta,\phi)$$

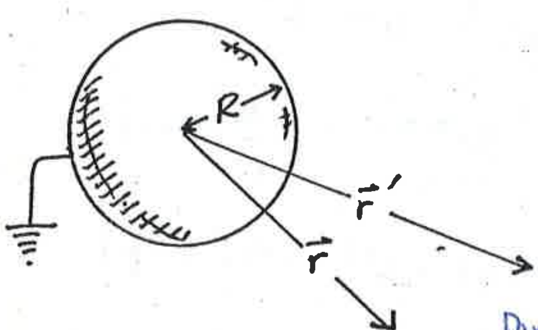
$$= \int_{-1}^{1} d(\cos\theta')\int_0^{2\pi} d\phi'\, f(\theta',\phi')\, \delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi')$$

WHENCE WE IDENTIFY

$$\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

THIS IS A DIRECT CONSEQUENCE OF THE COMPLETENESS PROPERTY OF THE Y_{lm}, AND SO IS CALLED THE completeness relation. SIMILAR RELATIONS HOLD FOR ALL COMPLETE SETS OF FUNCTIONS. WE'LL USE THIS IN TRYING TO FIND A CERTAIN SPECIAL QUANTITY.

RATHER THAN TRYING TO USE OUR COMPLETELY GENERAL EXPRESSION FOR $\varphi(\vec{r})$ AS A SOLUTION TO $\nabla^2\varphi = 0$, LET'S USE THE COMPLETENESS PROPERTY OF THE Y_{lm} TO FIND THE *Green's function* FOR A GROUNDED CONDUCTING SPHERE OF RADIUS R. WE'VE ALREADY DONE THIS (p. 28) FOR THE SPHERE USING THE BLACK MAGIC OF THE METHOD OF IMAGES: NOW WE'LL CALCULATE $G(\vec{r},\vec{r}\,')$ *a priori*. By SO DOING WE'LL ILLUSTRATE HOW YOU CAN CALCULATE $G(\vec{r},\vec{r}\,')$ FOR MOST ANY SYSTEM, EVEN IF YOU DON'T HAVE AN "EDUCATED GUESS" TO START WITH.



THE GREEN'S FUNCTION FOR THE EXTERIOR OF THE GROUNDED SPHERE SATISFIES

$$\nabla_r^2 G(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,'),$$

WITH BOUNDARY CONDITIONS

Dirichlet Boundary Conditions

(i) $G(\vec{r},\vec{r}\,') = 0$ FOR $|\vec{r}| = R$,

(ii) $G(\vec{r},\vec{r}\,') \to 0$ AS $|\vec{r}| \to \infty$.

BECAUSE OF THE COMPLETENESS OF THE SPHERICAL HARMONICS Y_{lm}, WE CAN WRITE $G(\vec{r}\,\vec{r}\,')$ is a function of θ, and ϕ, so we can write it as

$$G(\vec{r},\vec{r}\,') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}(r;\, r',\theta',\phi')\, Y_{lm}(\theta,\phi)$$

coefficient depends on r, r', θ', and ϕ'

SO THAT THE DIFFERENTIAL EQUATION GIVES

$$\nabla_r^2 G(\vec{r},\vec{r}\,') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\, a_{lm}(r;r',\theta',\phi')\right) - \frac{l(l+1)}{r^2}a_{lm}(r;r',\theta',\phi')\right]Y_{lm}(\theta,\phi)$$

$$= -4\pi\cdot\frac{\delta(r-r')}{r^2}\cdot\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi')$$

$$= -\frac{4\pi}{r^2}\delta(r-r')\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\phi')\, Y_{\ell m}(\theta,\phi)$$

BY THE COMPLETENESS PROPERTY.

THUS THE DEPENDENCE OF $a_{\ell m}(r; r', \theta', \phi')$ ON θ' AND ϕ' IS JUST THAT OF $Y^*_{\ell m}(\theta', \phi')$, AND WE WRITE

$$a_{\ell m}(r; r', \theta', \phi') = g_\ell(r, r')\, Y^*_{\ell m}(\theta', \phi')$$

WHERE THE radial Green's function $g_\ell(r, r')$ SATISFIES

$$r\frac{d^2}{dr^2}\left(r g_\ell(r,r')\right) - \ell(\ell+1)\, g_\ell(r,r') = -4\pi\delta(r-r').$$

For $r \neq r'$, THIS EQUATION IS JUST THE ONE WE SAW IN SOLVING THE RADIAL PART OF LAPLACE'S EQUATION (indeed, for $\vec{r} \neq \vec{r}'$, $G(\vec{r}, \vec{r}')$ satisfies Laplace's equation). BECAUSE OF THE δ-FUNCTION, HOWEVER, WE CANNOT EXPECT THE SOLUTION FOR $r < r'$ TO BE THE SAME AS FOR $r > r'$: FURTHERMORE, THE BOUNDARY CONDITIONS MANDATE THAT THE SOLUTIONS IN THE TWO REGIMES ARE DIFFERENT:

For $r < r'$, $$g_\ell(r, r') = A_\ell r^\ell + B_\ell r^{-(\ell+1)}$$
$$= A_\ell\left(r^\ell - \frac{R^{2\ell+1}}{r^{\ell+1}}\right)$$

These coefficients $A_\ell, B_\ell, C_\ell, D_\ell$ are functions of r', see page 41.

SINCE $g_\ell(r, r') = 0$ AT $r = R$.

For $r > r'$, $$g_\ell(r, r') = C_\ell r^\ell + D_\ell r^{-(\ell+1)}$$
$$= D_\ell r^{-(\ell+1)}$$

SINCE $g_\ell(r, r') \to 0$ AS $r \to \infty$

WE STILL MUST DETERMINE A_ℓ AND D_ℓ. TO DO THIS, WE NOTE THAT

(i) $g_\ell(r,r')$ MUST BE CONTINUOUS IN $\vec{r}$ BECAUSE $\varphi(\vec{r}) = \int d^3r'\, G(\vec{r},\vec{r}\,')\rho(\vec{r}\,')$ IS CONTINUOUS IN $\vec{r}$; AND

(ii) FROM THE DIFFERENTIAL EQUATION FOR $g_\ell(r,r')$, BY INTEGRATING FROM $r-\varepsilon$ TO $r+\varepsilon$ WE GET

$$\frac{d}{dr}\left(r g_\ell(r,r')\right)_{r=r'+\varepsilon} - \frac{d}{dr}\left(r g_\ell(r,r')\right)_{r=r'-\varepsilon} = -\frac{4\pi}{r'}.$$

(slope of g_ℓ is discontinuous)

THE FIRST CONDITION TELLS US THAT THE TWO SOLUTIONS MATCH AT $r=r'$, SO $D_\ell = A_\ell\left(r'^{2\ell+1} - R^{2\ell+1}\right)$; THE SECOND CONDITION THEN GIVES $A_\ell = \frac{4\pi}{(2\ell+1)r'^{\ell+1}}$.

WHEN ALL THE DUST SETTLES, THE RESULT FOR $G(\vec{r},\vec{r}\,')$ IS

$$G(\vec{r},\vec{r}\,') = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} g_\ell(r,r')\, Y^*_{\ell m}(\theta',\phi')\, Y_{\ell m}(\theta,\phi)$$

$$= 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1}\left[\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{R^{2\ell+1}}{(rr')^{\ell+1}}\right] \otimes Y^*_{\ell m}(\theta',\phi')\, Y_{\ell m}(\theta,\phi)$$

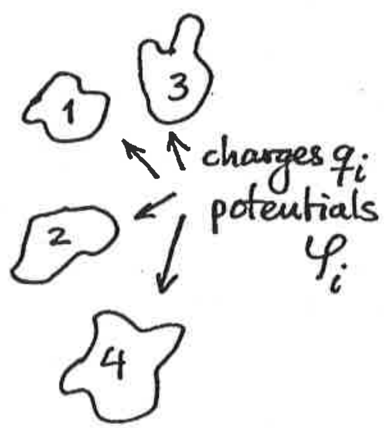
WHERE $r_<$ = THE LESSER OF r, r'

$r_>$ = THE GREATER OF r, r'.

THE ENTIRE EXPRESSION IS MANIFESTLY SYMMETRIC UNDER $\vec{r} \leftrightarrow \vec{r}'$. IT'S A STRAIGHTFORWARD EXERCISE TO SHOW THAT THIS IS JUST THE SAME AS THE EXPRESSION FOR $G(\vec{r},\vec{r}')$ FOUND FOR THE SPHERE ON P. 28 BY THE METHOD OF IMAGES.

THUS WE HAVE A *general* METHOD FOR FINDING $G(\vec{r},\vec{r}')$ ... WITH OTHER GEOMETRIES, OTHER COÖRDINATE SYSTEMS MAY BE MORE APPROPRIATE.

LET'S CONCLUDE OUR DISCUSSION OF THE ELECTROSTATICS OF CONDUCTORS BY CALCULATING THE ENERGY OF A SYSTEM OF CHARGED CONDUCTORS AND THE FORCES ON CONDUCTORS IN SUCH A SYSTEM. ALONG THE WAY WE WILL FIND A NONTRIVIAL RELATIONSHIP BETWEEN THE CHARGE ON A GIVEN CONDUCTOR AND THE POTENTIAL OF THAT AND ALL OTHER CONDUCTORS. WE BEGIN WITH THE EXPRESSION FOR THE TOTAL ENERGY OF THE ELECTROSTATIC FIELD OF THE CHARGED CONDUCTORS:



$$U = \frac{1}{8\pi}\int_V d^3r\, |\vec{E}|^2$$

THE VOLUME V SHOULD EXTEND OVER ALL SPACE — IN FACT IT NEED EXTEND ONLY OVER THE VOLUME EXTERNAL TO THE CONDUCTORS, SINCE $\vec{E}=0$ INSIDE EACH CONDUCTOR. THEN, USING $\vec{E} = -\vec{\nabla}\varphi$,

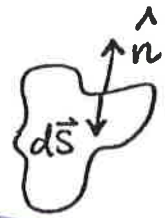
$$U = -\frac{1}{8\pi}\int_V d^3r\, \vec{E}\cdot\vec{\nabla}\varphi$$

$$= -\frac{1}{8\pi}\int_V d^3r \left[\vec{\nabla}\cdot(\vec{E}\varphi) - \varphi\underbrace{(\vec{\nabla}\cdot\vec{E})}_{=0}\right]$$

0 AS VOLUME V ENCLOSES NO CHARGES

$$= -\frac{1}{8\pi}\int_S d\vec{S}\cdot\vec{E}\varphi,$$

WHERE $d\vec{S}$ POINTS <u>OUT</u> OF THE VOLUME V. THE CONTRIBUTION FROM THE SURFACE AT INFINITY VANISHES, LEAVING JUST THE CONTRIBUTIONS FROM THE SURFACES OF THE CONDUCTORS. THEN



$\phi \sim 1/r$
$E \sim 1/r^2$
$ds \sim r^2$
so at ∞
$\int \sim 1/r \to 0$

$$U = -\frac{1}{8\pi} \underbrace{\sum_i}_{\text{sum over conductors}} \int_{S_i} (-\hat{n}\,da) \cdot \vec{E}\,\varphi$$

AS $d\vec{S}$ POINTS <u>into</u> EACH CONDUCTOR

$\hat{n}$ is normal out of conductors

$$= \frac{1}{8\pi} \sum_i \varphi_i \int_{S_i} E_n\,da$$

AS SURFACE IS EQUIPOTENTIAL

$$= \frac{1}{8\pi} \sum_i \varphi_i \,(4\pi q_i)$$

BY GAUSS' LAW (charge enclosed in S_i)

$$= \frac{1}{2} \sum_i q_i \varphi_i$$

(analogous to expression for a set of point charges)

NOW, THE q_i AND φ_i CANNOT BOTH BE GIVEN ARBITRARY VALUES — THERE ARE RELATIONS BETWEEN THEM. SINCE THE EQUATIONS OF ELECTROSTATICS ARE LINEAR, WE CAN READILY DETERMINE THESE RELATIONS. SUPPOSE $q_1 \neq 0$ BUT $q_i = 0$ FOR ALL $i \neq 1$; THEN THE POTENTIAL ON EACH CONDUCTOR MUST SIMPLY BE PROPORTIONAL TO q_1, SO THAT

$$\varphi_i = P_{i1}\, q_1 .$$

SIMILARLY, IF WE HAD $q_2 \neq 0$ BUT $q_i = 0$ FOR ALL $i \neq 2$, THE POTENTIAL ON EACH CONDUCTOR WOULD BE

$$\varphi_i = P_{i2}\, q_2$$

AND SO ON.

BECAUSE OF THE LINEARITY OF ELECTROSTATICS, IT FOLLOWS

THAT FOR AN ARBITRARY SET $\{q_1, q_2, \dots\}$ THE POTENTIALS ON THE CONDUCTORS ARE

$$\varphi_i = \sum_j P_{ij}\, q_j .$$

THIS IS A MATRIX RELATION — IT IS INVERTIBLE TO

$$q_i = \sum_j C_{ij}\, \varphi_j .$$

THE MATRIX C_{ij} IS CALLED THE *capacitance tensor*. ITS ELEMENTS HAVE DIMENSIONS OF LENGTH AND DEPEND ON THE SHAPE AND RELATIVE POSITION OF THE CONDUCTORS.

lengths present are sizes of conductors and their separations

IF THERE IS ONLY ONE CONDUCTOR, THE SINGLE ELEMENT C IS CALLED THE *capacity*:

$$q = C\varphi$$

WHICH SAYS HOW MUCH CHARGE THE CONDUCTOR WILL HOLD AT A GIVEN POTENTIAL. C IS JUST A MEASURE OF THE SIZE OF THE CONDUCTOR. FOR EXAMPLE, FOR A SPHERE OF RADIUS R CARRYING CHARGE Q, THE POTENTIAL IS

The only length around is size of conductor

$$\varphi(\vec{r}) = \begin{cases} Q/r, & r \geq R \\ Q/R, & r < R \end{cases}$$

potential inside is constant.

SO THAT THE POTENTIAL OF THE CONDUCTOR ITSELF IS Q/R AND

$$C = Q/\varphi = R,$$ THE RADIUS OF THE CONDUCTOR.

WE CAN FIND A USEFUL RELATION BETWEEN THE C_{ij} BY RETURNING TO OUR EXPRESSION FOR THE ENERGY U.
WE'LL DO THIS BY CALCULATING THE *change* IN THE ENERGY U OF A SYSTEM OF CONDUCTORS CAUSED BY A CHANGE IN THEIR CHARGES OR POTENTIALS.

FROM THE ORIGINAL EXPRESSION

$$U=\frac{1}{8\pi}\int d^3r\,|\vec{E}|^2,$$

A CHANGE IN THE q_i OR φ_i WILL GIVE RISE TO A CHANGE IN $\vec{E}$ AND SO A CHANGE IN U:

$$\delta U=\frac{1}{8\pi}\int d^3r\ \delta|\vec{E}|^2 = \frac{1}{4\pi}\int d^3r\ \vec{E}\cdot\delta\vec{E}$$

LET'S WORK THIS OUT FOR THE CASE WHERE WE VARY q_i AND WHERE WE VARY φ_i.

(i) CHANGE charges: THEN $q_i\to q_i+\delta q_i$. IN MAKING THIS CHANGE WE DO WORK, SINCE WE ARE MOVING CHARGES IN THE PRESENCE OF FIXED POTENTIALS. WE WRITE, USING $\vec{E}=-\vec{\nabla}\varphi$,

$$\delta U=\frac{1}{4\pi}\int d^3r\,\vec{E}\cdot\delta\vec{E} = -\frac{1}{4\pi}\int d^3r\ \vec{\nabla}\varphi\cdot\delta\vec{E}$$

$$= -\frac{1}{4\pi}\int d^3r\ \vec{\nabla}\cdot(\varphi\,\delta\vec{E}) + \frac{1}{4\pi}\int d^3r\ \varphi(\vec{\nabla}\cdot\delta\vec{E})$$

Now, $\vec{\nabla}\cdot\vec{E}=0$ (our integral is only over the exterior of the conductors), AND LIKEWISE $\vec{\nabla}\cdot(\vec{E}+\delta\vec{E})=0$ (the new electric field also satisfies Maxwell's eqn.) so THAT $\vec{\nabla}\cdot\delta\vec{E}=0$ AND

because $\vec{E}=0$ within conductor

$$\delta U = -\frac{1}{4\pi}\int d^3r\ \vec{\nabla}\cdot(\varphi\,\delta\vec{E})$$

$$=\frac{1}{4\pi}\sum_i \varphi_i\int_{S_i}\delta E_n\,da$$

sum over conductors

BY THE DIVERGENCE THM: WE GET AN EXTRA MINUS SIGN BECAUSE $\hat{n}$ POINTS into THE VOLUME OF INTEGRATION, NOT OUT (viz. p. 43)

φ_i is constant on ith conductor & can be taken out of integral

$$=\frac{1}{4\pi}\sum_i\varphi_i\cdot 4\pi\,\delta q_i$$

$$=\sum_i\varphi_i\,\delta q_i$$

IS THE CHANGE IN ENERGY FROM VARYING THE CHARGES. THIS MAKES SENSE: $\varphi_i\,\delta q_i$ IS JUST THE ENERGY NEEDED

TO MOVE A CHARGE δq_i FROM INFINITY (where $\varphi=0$) TO THE iTH CONDUCTOR (WHERE $\varphi=\varphi_i$).

(ii) CHANGE potentials: THEN $\varphi_i \to \varphi_i + \delta\varphi_i$. THEN THE NEW ELECTRIC FIELD IS

$$\vec{E}+\delta\vec{E} = -\vec{\nabla}(\varphi+\delta\varphi), \text{ so that } \delta\vec{E} = -\vec{\nabla}\delta\varphi$$

AND

$$\delta U = \frac{1}{4\pi}\int d^3r\, \vec{E}\cdot\delta\vec{E} = -\frac{1}{4\pi}\int d^3r\,(\vec{E}\cdot\vec{\nabla})\delta\varphi$$

$$= -\frac{1}{4\pi}\int d^3r\; \vec{\nabla}\cdot(\vec{E}\,\delta\varphi) + \frac{1}{4\pi}\int d^3r\,\underbrace{(\vec{\nabla}\cdot\vec{E})}_{=0}\,\delta\varphi$$

$$= \frac{1}{4\pi}\sum_i \delta\varphi_i \int_{S_i} E_n\, da \quad \text{BY THE DIVERGENCE THEOREM}$$

$$= \frac{1}{4\pi}\sum_i \delta\varphi_i \cdot (4\pi q_i)$$

$$= \sum_i q_i\, \delta\varphi_i,$$ WHICH IS THE CHANGE IN ENERGY FROM VARYING THE POTENTIALS.

THUS WE HAVE

(i) $\delta U = \sum_i \varphi_i\, \delta q_i$ IF WE TREAT THE q_i AS THE INDEPENDENT VARIABLES; AND

(ii) $\delta U = \sum_i q_i\, \delta\varphi_i$ IF WE TREAT THE φ_i AS THE INDEPENDENT VARIABLES.

FROM THESE RESULTS WE SEE THAT

$$\frac{\partial U}{\partial q_i} = \varphi_i \quad \text{AND} \quad \frac{\partial U}{\partial \varphi_i} = q_i\ ;$$

BUT THE q_i AND φ_i ARE LINEAR FUNCTIONS OF EACH OTHER, SINCE $q_i = \sum_j C_{ij}\,\varphi_j$. THUS

$$\frac{\partial}{\partial\varphi_i}\left(\frac{\partial U}{\partial\varphi_j}\right) = \frac{\partial}{\partial\varphi_i}(q_j)$$

$$= \frac{\partial}{\partial\varphi_i}\sum_k C_{jk}\varphi_k$$

$$= C_{ji}\ ;$$

BUT, REVERSING THE ORDER OF DIFFERENTIATION, WE GET

$$\frac{\partial}{\partial\varphi_j}\left(\frac{\partial U}{\partial\varphi_i}\right) = \frac{\partial}{\partial\varphi_j}(q_i)$$

$$= \frac{\partial}{\partial\varphi_j}\sum_k C_{ik}\varphi_k$$

$$= C_{ij}.$$

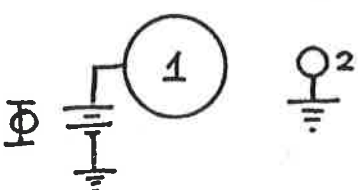
CLEARLY $\dfrac{\partial^2 U}{\partial\varphi_i\partial\varphi_j} = \dfrac{\partial^2 U}{\partial\varphi_j\partial\varphi_i}$, SO

U, $\frac{\partial U}{\partial\varphi_j}$, $\frac{\partial U}{\partial\varphi_i}$, $\frac{\partial^2 U}{\partial\varphi_j\partial\varphi_i}$, $\frac{\partial^2 U}{\partial\varphi_i\partial\varphi_j}$ all continuous so this result follows

$C_{ij} = C_{ji}$, I.E. THE CAPACITANCE TENSOR OF *any* SYSTEM OF CONDUCTORS IS *symmetric!*

THIS IS A *nontrivial* RESULT. CONSIDER TWO CONDUCTORS AS AN EXAMPLE:

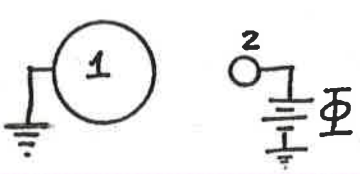
SITUATION (a): #1 AT POTENTIAL Φ, #2 GROUNDED —



THEN $q_1 = C_{11}\Phi$, $q_2 = C_{21}\Phi$

charge on grounded conductor

SITUATION (b): #1 GROUNDED, #2 AT POTENTIAL Φ —



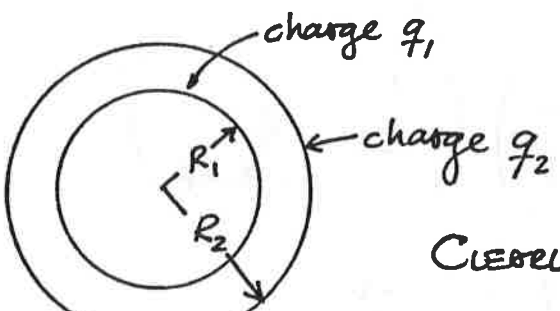
THEN $q_1 = C_{12}\Phi$, $q_2 = C_{22}\Phi$

charge on grounded conductor

even though $C_{11}\Phi > C_{22}\Phi$ so direct charges not the same!

SINCE $C_{12} = C_{21}$, THE same charge IS INDUCED ON THE GROUNDED CONDUCTOR IN THE TWO SITUATIONS, independent OF THE SHAPE OR SIZE OF THE TWO CONDUCTORS.

WE CAN SEE THE SYMMETRY OF THE CAPACITANCE TENSOR BY EXPLICIT EXAMPLE FOR A SIMPLE CASE. CONSIDER A spherical capacitor: THIS IS TWO SPHERICAL CONDUCTING SHELLS ARRANGED AS SHOWN. WITH CHARGES q_1 AND q_2 AS SHOWN, TO FIND THE C_{ij} WE NEED TO KNOW φ_1 AND φ_2.



CLEARLY $\varphi(\vec{r})$ IS INDEPENDENT OF ANGLE, SO

$$\varphi(\vec{r}) = \varphi(r) = \begin{cases} A/r, & r \geq R_2 \\ B + C/r, & R_2 \geq r \geq R_1 \\ D, & R_1 \geq r \end{cases}$$

CLEARLY $A = q_1 + q_2$ BY GAUSS' LAW AND $C = q_1$ BY GAUSS' LAW: THESE FOLLOW SINCE $\vec{E} = \hat{r}(q_1+q_2)/r^2$ FOR $r > R_2$ AND $\vec{E} = \hat{r} q_1/r^2$ FOR $R_2 > r > R_1$. SINCE $\varphi(r)$ IS CONTINUOUS,

$$\frac{A}{R_2} = B + \frac{C}{R_2} \quad (\text{match at } r = R_2)$$

$$\Longrightarrow B = q_2/R_2$$

$$D = B + \frac{C}{R_1} \quad (\text{match at } r = R_1)$$

$$\Longrightarrow D = \frac{q_1}{R_1} + \frac{q_2}{R_2}$$

WE SEE THAT

$$\varphi_1 = \varphi(r=R_1) = \frac{q_1}{R_1} + \frac{q_2}{R_2}$$

$$\varphi_2 = \varphi(r=R_2) = \frac{q_1}{R_2} + \frac{q_2}{R_2}$$

WHICH GIVES US φ_i IN TERMS OF THE q_i. INVERTING, WE GET

$$q_1 = \left(\frac{R_1R_2}{R_2-R_1}\right)\varphi_1 - \left(\frac{R_1R_2}{R_2-R_1}\right)\varphi_2$$

$$q_2 = -\left(\frac{R_1R_2}{R_2-R_1}\right)\varphi_1 + \left(\frac{R_2^2}{R_2-R_1}\right)\varphi_2$$

SO THAT

$$C_{11} = \frac{R_1R_2}{R_2-R_1} > 0$$

$$C_{12} = C_{21} = -\left(\frac{R_1R_2}{R_2-R_1}\right) < 0$$

$$C_{22} = \frac{R_2^2}{R_2-R_1} > 0$$

THE TENSOR C_{ij} IS MANIFESTLY SYMMETRIC. IT IS GENERALLY TRUE THAT

$C_{ii} > 0$ AS A POSITIVE POTENTIAL ON i MEANS A POSITIVE CHARGE ON i;

expect this physically → $C_{ij} < 0$ $(i \neq j)$ AS A POSITIVE POTENTIAL ON j INDUCES A NEGATIVE CHARGE ON i.

WE NOTE THAT THE *usual* "CAPACITANCE" WE TALK ABOUT FOR A CAPACITOR IS THE CHARGE HELD ON ONE OF THE PLATES IF ONE PLATE IS GROUNDED AND THE OTHER PLATE IS AT UNIT POTENTIAL. FOR THIS SPHERICAL CAPACITOR, IF WE TAKE

i.e. $C = \frac{\Delta Q}{\Delta V}$

$$\varphi_2 = 0, \quad \varphi_1 = V, \quad \text{THEN}$$

$$q_1 = \left(\frac{R_1R_2}{R_2-R_1}\right)V, \qquad q_2 = -q_1,$$

AND THE CAPACITANCE IS $$C = \frac{q_1}{V} = \frac{R_1R_2}{R_2-R_1}.$$

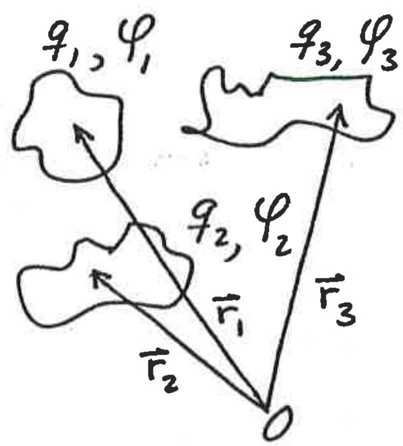
JUST AS FOR AN ORDINARY PARALLEL-PLATE CAPACITOR, THE CAPACITANCE INCREASES AS THE SURFACES OF THE CAPACITOR ARE BROUGHT CLOSER TOGETHER.

WE FINALLY NOTE THAT, USING $q_i = \sum_j C_{ij}\varphi_j$, THE TOTAL ENERGY OF A SYSTEM OF CONDUCTORS IS

$$U = \frac{1}{2}\sum_i q_i \varphi_i = \frac{1}{2}\sum_{i,j} C_{ij}\varphi_i\varphi_j \geq 0$$

U IS POSITIVE DEFINITE BECAUSE IT IS JUST $\frac{1}{8\pi}\int d^3r\, |\vec{E}|^2 \geq 0$; THE SELF-ENERGY OF EACH CONDUCTOR IS INCLUDED.

FINALLY, LET'S CONSIDER HOW TO CALCULATE THE *forces* ACTING ON EACH CONDUCTOR IN A SYSTEM OF CHARGED CONDUCTORS.



IN GENERAL THE iTH CONDUCTOR WILL FEEL AN ELECTROSTATIC FORCE $\vec{F}_i$ DUE TO THE CHARGES ON ALL OTHER CONDUCTORS. IN ORDER THAT THIS CONDUCTOR REMAIN FIXED, WE (THE EXTERNAL AGENT) MUST EXERT A FORCE

$$\vec{F}_{ext_i} = -\vec{F}_i \quad \text{ON THE } i\text{TH CONDUCTOR}$$

IF WE THEN MOVE THE iTH CONDUCTOR BY A SMALL AMOUNT $\delta\vec{r}_i$, WE DO AN AMOUNT OF MECHANICAL WORK $\vec{F}_{ext_i}\cdot\delta\vec{r}_i$; THE TOTAL MECHANICAL WORK DONE FOR SMALL DISPLACEMENTS OF ALL THE CONDUCTORS IS THEN

$$W_{mech} = \sum_i \vec{F}_{ext_i}\cdot\delta\vec{r}_i = -\sum_i \vec{F}_i\cdot\delta\vec{r}_i$$

electrostatic force on iTH conductor

if we move conductor infinitely slowly