THE CENTERPIECE OF THE COURSE, WHICH WE WILL BE TRYING TO UNDERSTAND IN THEIR RAMIFICATIONS, IS THE SET OF MAXWELL'S EQUATIONS:

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \rightarrow -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\cdot\vec{E} = 4\pi\rho \rightarrow \rho/\epsilon_0$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\rightarrow \frac{1}{c^2}\partial E/\partial t + \mu_0\vec{J}$$

- HERE $\vec{E}$ IS THE electric field (STATVOLTS CM^{-1})
  - $\vec{B}$ IS THE magnetic induction
  - ρ IS THE electric charge density (COULOMBS CM^{-3})
  - $\vec{J}$ IS THE electric current density (AMPERES CM^{-2})
  - c IS THE SPEED OF LIGHT IN VACUUM ($CM\,S^{-1}$)

WE NOTE THAT ELECTRIC CHARGE IS KNOWN TO BE conserved SO THAT $\vec{J}$ AND ρ ARE NOT INDEPENDENT BUT RELATED BY THE CONTINUITY EQUATION $\vec{\nabla}\cdot\vec{J} + \partial\rho/\partial t = 0$. →

- AS WE WILL SEE LATER ON, THESE EQUATIONS ARE MODIFIED SOMEWHAT IN MEDIA, BUT THE ESSENTIAL PHYSICS IS AS EMBODIED IN MAXWELL'S EQUATIONS AS GIVEN ABOVE.

FOR MUCH OF THE COURSE WE WILL CONSIDER SOURCES AND FIELDS WHICH ARE static IN TIME. THEN $\vec{E}$ AND $\vec{B}$ BECOME DECOUPLED:

$$\vec{\nabla}\times\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

(The sources are still constrained by the continuity

CMI

Implicit in the statement of Maxwell's Eqns is that $\rho(\vec{r})$ and $J(\vec{r})$ are <u>known</u>.

More typically (at least in CMP) this is <u>not</u> the case!! Indeed, the <u>goal</u> is to compute $\rho(\vec{r})$ and $J(\vec{r})$!! A combined QM and EM problem

$$\hat{H} = \sum_{\ell} \hat{p}_e^2/2m + V(\{\vec{r}_e\})$$

$$\hat{H}\,\Psi(\vec{r}_1, \vec{r}_2 \ldots \vec{r}_N) = E\,\Psi(\vec{r}_1, \vec{r}_2 \ldots \vec{r}_N)$$

Many e^-

Assume nuclear positions R_L are <u>frozen</u>

(adiabatic approximation) $M_N \gg m_e$

$$V(\vec{r}_e) = -\sum_{\ell, L} kZe^2/|\vec{r}_e - \vec{R}_L| + \sum_{\ell\ell'} ke^2/|\vec{r}_e - \vec{r}_e'|$$

Already Hydrogen atom is hard, Bethe...

Q: Does anyone know what is done?

CM2.

$$V_{eff}(r) = -\sum_L kZe^2/|\vec{r}-\vec{R}_L|$$

↙ single particle!

$$+ ke^2 \sum_i \int d^3r' \, |\psi_i(r')|^2/|\vec{r}-\vec{r}'|$$

where $\psi_i(r')$ are solns of $\rho(r)$

$$H_{eff}\,\psi = E\psi$$

$$H_{eff} = p^2/2m + V_{eff}(r)$$

$\sum_i$ is over $N_{electron}$ lowest E states.

Self consistent: guess initial $\{\psi_i(r)\} \to$ iterate.

Why is this EM course useful? All the ~~tricks~~ calculational tools (Legendre poly expansions etc) in ~~computing~~ dealing with EM field are central to solving *.

↙ Notice we begin with known $\psi_i(r)$ → known ρ.

NOTE THAT $\vec{E}$ IS irrotational ($\vec{\nabla}\times\vec{E}=0$: ONLY TRUE IN STATICS) AND $\vec{B}$ IS solenoidal ($\vec{\nabla}\cdot\vec{B}=0$: ALWAYS TRUE)

... RECALL THAT IN GENERAL any VECTOR FIELD CAN BE SPLIT INTO AN IRROTATIONAL PART AND A SOLENOIDAL PART. →

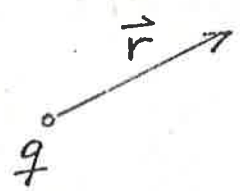
TO START WITH, WE'LL WORK WITH THE ELECTRIC FIELD $\vec{E}$ AND RETURN TO THE MAGNETIC INDUCTION $\vec{B}$ LATER IN THE COURSE. RECALL THAT $\vec{E}$ IS DEFINED IN TERMS OF THE FORCE $\vec{F}$ ON A CHARGE q AT POSITION $\vec{r}$:

$$\vec{F}(\vec{r}) = q\vec{E}(\vec{r}).$$

[electrostatic force due to other electric charges - depends on their charges & positions]

THE FIELD $\vec{E}$ IS ITSELF PRODUCED BY ELECTRIC CHARGES. WHAT IS THE ELECTRIC FIELD PRODUCED BY SOME ARRANGEMENT OF POINT CHARGES? FOR A single CHARGE q AT THE ORIGIN, COULOMB'S LAW SAYS THAT THE RESULTING FIELD $\vec{E}$ IS

$$\vec{E}(\vec{r}) = \frac{q}{r^2}\hat{r} = \frac{q}{r^3}\vec{r} \rightarrow \frac{Kq\hat{r}}{r^2}$$



THUS THE $\vec{E}$-FIELD FROM A POINT CHARGE IS PURELY radial (RADIAL out FOR $q>0$, RADIAL in FOR $q<0$).

Question: DOES THIS EXPRESSION FOR $\vec{E}$ SATISFY MAXWELL'S EQUATIONS? LET US WORK IN CARTESIAN COORDINATES:

$$\vec{E} = \hat{\imath}E_x + \hat{\jmath}E_y + \hat{k}E_z,$$

WITH $E_x = \hat{\imath}\cdot\vec{E} = \frac{q}{r^3}\vec{r}\cdot\hat{\imath} = \frac{qx}{r^3}$ AND SIMILARLY FOR E_y AND E_z.

(i) IS $\vec{\nabla}\times\vec{E}=0$ SATISFIED? THE z-COMPONENT

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NATIONAL

The study of electricity and magnetism is replete with vector identities. We will, for example, very shortly discuss the use of Gauss' Theorem to convert the differential form of the Maxwell eqn $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ to an integral version $\Phi_E = Q/\epsilon_0$

$$\Phi_E = \int \vec{E} \cdot \hat{n}\, dS$$

Some texts on E+M begin with a review of vector identities and vector calculus. This is awkward because it is (1) very dry and (2) pretty familiar to some of you.

However, let's take a short digression which I hope will be at least less familiar.

V-2

Vector identities and the Levi Civita symbol ϵ_{ijk}

You are all familiar with the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases} \qquad i,j = 1,2,3$$

The kronecker δ is symmetric under interchange of i,j and naturally associated with the <u>dot</u> product

$$\vec{a}\cdot\vec{b} = a_i b_i = \delta_{ij} a_i b_j \quad \leftarrow \text{(summation understood)}$$

The Levi-Civita symbol is <u>anti</u> symmetric under exchange of any two indices

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$$

$$\epsilon_{123} = 1 \qquad \epsilon_{122} = 0 = \epsilon_{233} = \cdots$$

$$\epsilon_{213} = -1 \text{ etc}$$

Q: It is naturally associated with the cross product

$$(a \times b)_i = \epsilon_{ijk} a_j b_k$$

summation understood

An important identity relating the LCS and KD is

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

(Convince yourself this is true!)

Then the LCS is very useful for proving vector identities, eg.

Q: $[\vec{a} \times (\vec{b} \times \vec{c})] = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$

proof $[\vec{a} \times (\vec{b} \times \vec{c})]_i$

$$= \epsilon_{ijk}\, a_j (b \times c)_k$$

$$= \epsilon_{ijk}\, a_j\, \epsilon_{klm}\, b_l c_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, a_j b_l c_m$$

$$= a_j b_i c_j - a_j b_j c_i$$

$$= [\vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})]_i$$

Useful also when one of the "vectors"

is the gradient $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

For example

Q. $\vec{\nabla} \times \vec{\nabla} f = ?$

$$(\vec{\nabla} \times \vec{\nabla} f)_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_k}$$

order of differentiation is irrelevant

$$= \epsilon_{ijk} \frac{\partial}{\partial x_k} \frac{\partial f}{\partial x_j}$$

rename indices $j \leftrightarrow k$

$$= \epsilon_{ikj} \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_k}$$

asym of LCS

$$= -\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_k}$$

$$\Rightarrow \phi,$$

Q: Does someone know a more immediate way to see this vanishes?
A! sym x asym

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \nabla^2 v$$

is an interesting exercise.

ASIDE

This index notation is also very useful

in linear algebra

$$(AB)_{ij} = A_{ik}B_{kj}$$ ← summation over k understood

Skip Q? $(AB)C =$

$$[(AB)C]_{ij} = (AB)_{ik}C_{kj}$$
$$= A_{i\ell}B_{\ell k}C_{kj}$$
$$= A_{i\ell}(BC)_{\ell j}$$
$$= [A(BC)]_{ij}$$

Q $(AB)^T =$

$$(AB)^T{}_{ij} = (AB)_{ji}$$
$$= A_{jk}B_{ki}$$
$$= B_{ki}A_{jk} = (B^T)_{ik}(A^T)_{kj}$$
$$= (B^TA^T)_{ij}$$

V-6

Proof that any vector field can be written as sum of irrotational ($\vec{\nabla}\times\vec{A}=0$) and solenoidal ($\vec{\nabla}\cdot\vec{A}=0$) parts

Fourier decompose

$$\vec{A}(\vec{r}) = \int d^3\vec{k}\; \vec{A}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$$

and write
$$\vec{A}(\vec{k}) = \underbrace{\vec{k}\cdot\vec{A}(\vec{k})\,\frac{\vec{k}}{k^2}}_{\text{"}\vec{A}_L(\vec{k})\text{"}} + \underbrace{\vec{A}(\vec{k}) - \vec{k}\cdot\vec{A}(\vec{k})\,\frac{\vec{k}}{k^2}}_{\text{"}\vec{A}_T(\vec{k})\text{"}}$$

longitudinal, the part of $\vec{A}$ lying along $\vec{k}$

obviously $\vec{A}(\vec{r}) = \vec{C}(\vec{r}) + \vec{B}(\vec{r})$

where
$$\vec{C}(\vec{r}) \equiv \int d^3k\; \vec{A}_L(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$$
$$\vec{B}(\vec{r}) \equiv \int d^3k\; A_T(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$$

But,...
$$\vec{\nabla}\cdot\vec{B} = \int d^3k\; \underbrace{\vec{A}_T(\vec{k})\cdot i\vec{k}}_{\phi\,!}\; e^{i\vec{k}\cdot\vec{r}}$$

and
$$\vec{\nabla}\times\vec{C} = \int d^3k\; \underbrace{\vec{A}_L(\vec{k})\times i\vec{k}}_{\phi\,!}\; e^{i\vec{k}\cdot\vec{r}}$$

$$(\vec{\nabla}\times\vec{E})_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}$$

$$= \frac{\partial}{\partial x}\left(\frac{qy}{r^3}\right) - \frac{\partial}{\partial y}\left(\frac{qx}{r^3}\right)$$

$$= \frac{-3qy}{r^4}\frac{\partial r}{\partial x} + \frac{3qx}{r^4}\frac{\partial r}{\partial y} \quad \text{AS} \quad \frac{\partial y}{\partial x} = \frac{\partial x}{\partial y} = 0.$$

NOW, $r = \sqrt{x^2+y^2+z^2}$ ACCORDING TO MR. PYTHAGORAS,

SO THAT

$$\frac{\partial r}{\partial x} = \frac{\frac{1}{2}\cdot 2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}, \quad \text{AND SIMILARLY} \quad \frac{\partial r}{\partial y} = \frac{y}{r}.$$

$$\left[\text{in general: } \vec{\nabla} r = \frac{\vec{r}}{r} = \hat{r}\right]$$

THUS THE z-COMPONENT OF THE CURL BECOMES

$$(\vec{\nabla}\times\vec{E})_z = -\frac{3qy}{r^4}\cdot\frac{x}{r} + \frac{3qx}{r^4}\cdot\frac{y}{r} = 0.$$

SIMILARLY $(\vec{\nabla}\times\vec{E})_x = (\vec{\nabla}\times\vec{E})_y = 0$,

SO THAT $\vec{\nabla}\times\vec{E} = 0$ AND $\vec{E}$ IS IRROTATIONAL

AS DESIRED... AS IT MUST BE FOR A <u>CENTRAL</u> FORCE.

(ii) IS $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ SATISFIED? THE DIVERGENCE IS

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{qx}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{qy}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{qz}{r^3}\right)$$

$$= q\left(\frac{1}{r^3} - \frac{3x}{r^4}\frac{\partial r}{\partial x}\right) + (y\text{-term}) + (z\text{-term})$$

$$= q\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) + (y\text{-term}) + (z\text{-term})$$

$$= \frac{3q}{r^3} - \frac{3q}{r^5}(x^2+y^2+z^2) = \frac{3q}{r^3} - \frac{3q}{r^3} = 0$$

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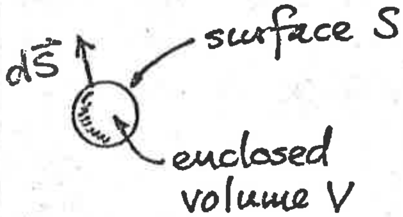
THUS $\vec{\nabla}\cdot\vec{E}=0$, WHICH IS OK WHEREVER $\rho=0$. IN OUR CASE $\rho=0$ EVERYWHERE EXCEPT AT $\vec{r}=0$: INDEED, OUR CALCULATION OF THE DIVERGENCE OF $\vec{E}$ IS ONLY VALID FOR $\vec{r}\neq 0$, SINCE AT $\vec{r}=0$, $\vec{E}=q\vec{r}/r^3$ IS NOT DEFINED. SO WE HAVE YET TO SHOW THAT $\vec{\nabla}\cdot\vec{E}=4\pi\rho$ IS SATISFIED AT $\vec{r}=0$, WHERE THE CHARGE IS LOCATED.

TO CONSIDER WHAT HAPPENS NEAR $\vec{r}=0$, CONSIDER A SMALL SPHERE AROUND THE CHARGE OF RADIUS a.



BY GAUSS' THEOREM,

$$\int_V d^3r\,\vec{\nabla}\cdot\vec{E} = \int_S d\vec{S}\cdot\vec{E}$$

(valid for any vector field $\vec{E}$).

FOR A SPHERE, THE R.H.S. OF THIS IS

$$\int_S d\vec{S}\cdot\vec{E} = \int_{4\pi} d\Omega\cdot a^2\,\hat{r}\cdot\vec{E}\Big|_{r=a} = \int_{4\pi} d\Omega\cdot a^2\cdot\frac{q}{a^2} = 4\pi q\,;$$

THIS IS THE FAMILIAR GAUSS' LAW.

THUS

$$\int_{r<a} d^3r\,\vec{\nabla}\cdot\vec{E} = 4\pi q, \text{ independent of the value of } a!$$

THIS CURIOUS RESULT IS DUE TO $\vec{E}$ BEING SINGULAR AT THE ORIGIN. TO ACCOUNT FOR THIS, WE WRITE

$$\vec{\nabla}\cdot\vec{E} = 4\pi q\,\delta(\vec{r}) = 4\pi q\,\delta(x)\delta(y)\delta(z),$$

WHERE THE *Dirac delta function* $\delta(x)$ SATISFIES

$$\int_a^b dx\,\delta(x) = \begin{cases} 1 & \text{IF } [a,b] \text{ INCLUDES } x=0 \\ 0 & \text{OTHERWISE.} \end{cases}$$

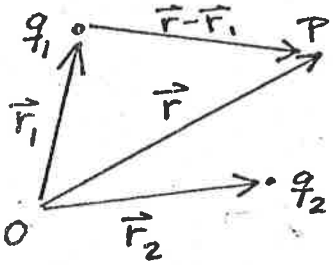
[$\delta(x)$ has units L^{-1}]

THUS $\vec{\nabla}\cdot\vec{E}=0$ EXCEPT AT THE ORIGIN, WHERE THE CHARGE

$$\rho(\vec{r}) = q\delta(\vec{r}).$$

THE CHARGE DENSITY VANISHES EXCEPT AT THE ORIGIN, WHERE IT IS SINGULAR.

THUS WE CONCLUDE THAT $\vec{E} = q\vec{r}/r^3$ SATISFIES MAXWELL'S EQUATIONS. SINCE MAXWELL'S EQUATIONS FOR $\vec{E}$ ARE linear IN THE CHARGE DENSITY ρ, THE SUPERPOSITION PRINCIPLE APPLIES AND THE FIELD $\vec{E}$ DUE TO SEVERAL CHARGES IS THE SUM OF THE FIELDS DUE TO THE INDIVIDUAL CHARGES:

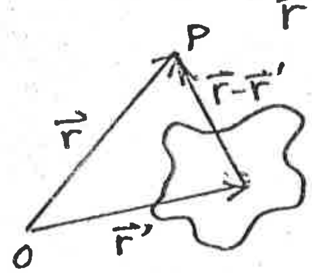


$$\vec{E}(\vec{r}) = \sum_i \frac{q_i(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3}$$

FOR A continuous CHARGE DISTRIBUTION, WE REPLACE q_i BY THE CHARGE $\rho(\vec{r}_i)\Delta V_i$ WITHIN A VOLUME ΔV_i AT $\vec{r}_i$: TAKING THE LIMIT $\Delta V_i \to 0$, WE GET

$$\vec{E}(\vec{r}) = \sum_i \Delta V_i \, \frac{\rho(\vec{r}_i)(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3} \longrightarrow \int d^3r' \, \frac{\rho(\vec{r}')(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

HERE $\vec{r}'$ = POSITION VECTOR OF CHARGE ELEMENT
$\vec{r}$ = POSITION VECTOR OF OBSERVATION POINT



<u>Check</u>: FOR discrete CHARGES,

$$\rho(\vec{r}') = \sum_i q_i \delta(\vec{r}'-\vec{r}_i),$$

SO THAT $\vec{E}(\vec{r})$ BECOMES

$$\vec{E}(\vec{r}) = \sum_i q_i \int d^3r' \, \frac{\delta(\vec{r}'-\vec{r}_i)\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$= \sum_i q_i (\vec{r}-\vec{r}_i)/|\vec{r}-\vec{r}_i|^3, \text{ AS BEFORE.}$$

THUS, IF WE KNOW $\rho(\vec{r})$ THROUGHOUT ALL SPACE, WE CAN FIND $\vec{E}(\vec{r})$ AT ANY POINT IN SPACE.

One of you inquired how the iterative method I outlined to get $\phi(x)$ in CMP could possibly work.

Answer with much much simpler example.

Suppose we wanted to solve d=1 Poisson Eqn

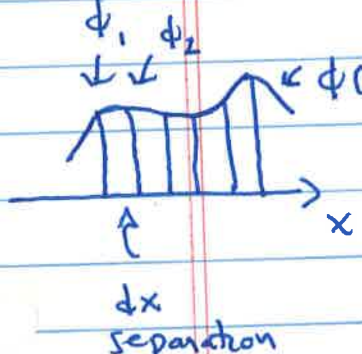
$$-\nabla^2\phi = \rho/\epsilon_0 \xrightarrow[d=1]{} -\frac{d^2\phi}{dx^2} = \rho(x)$$

"source" ρ/ϵ_0

Drop ϵ_0

Claim: Start with $\{\phi_i\}$

values of $\phi(x)$ on discrete set of points



old values

$$\phi_i' = \frac{1}{2}\left[\phi_{i+1} + \phi_{i-1} + (dx)^2\rho_i\right]$$

new values

in words average of potential @ neighbor pts + charge density

You will converge to soln of $-\frac{d^2\phi}{dx^2} = \rho(x)$!

Example $\rho(x) = 12x^2$ ← charge density

$0 < x < 1$ ← box

$\phi(0) = \phi(1) = 0$ ← bdy conditions

Soln is easy $d^2\phi/dx^2 = -12x^2$

$d\phi/dx = A - 4x^3$

$\phi = B + Ax - x^4$

$\phi(0) = B = 0 \qquad \phi(1) = 0 = A - 1$

Where did this eqn * come from and why doesn't it work?!

(A) Q: $\frac{d\phi}{dx} = ? = \frac{\phi_{i+1} - \phi_i}{dx}$

Q: $\frac{d^2\phi}{dx^2} = ? \quad \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{(dx)^2}$

So * is "derived" by

$$\phi_{i+1} - 2\phi_i + \phi_{i-1} = -(dx)^2 \rho_i$$

and "solving" for $\phi_i = \frac{1}{2}\left[\phi_{i+1} + \phi_{i-1} + (dx)^2 \rho_i\right]$

(B) If you define

$$\mathcal{L} = \frac{1}{2}\sum_j (\phi_{j+1} - \phi_j)^2 - (dx)^2 \sum \rho_j \phi_j$$

you can prove that $\phi_i \to \phi_i' = \frac{1}{2}\left[\phi_{i+1} + \phi_{i-1} + (dx)^2 \rho_i\right]$

SKIP ↓

always results in $\Delta \mathcal{L} < 0$!

So iterating * leads to a minimum of

$$\underbrace{\frac{1}{2}\int \left(\frac{d\phi}{dx}\right)^2 dx}_{\text{electric field energy}} \; - \underbrace{\int \rho\phi\, dx}_{\text{energy of charge } \rho \text{ in potential } \phi}$$

E

$\ominus$ sign??

P-3

$$du = \frac{1}{2}(\phi_{i+1} - \phi_i')^2 + \frac{1}{2}(\phi_i' - \phi_{i-1})^2 - dx^2 \rho_i \phi_i'$$

$$- \frac{1}{2}(\phi_{i+1} - \phi_i)^2 - \frac{1}{2}(\phi_i - \phi_{i-1})^2 + dx^2 \rho_i \phi_i$$

$$= \frac{1}{2}\Big\{ -2\phi_{i+1}(\phi_i' - \phi_i) + \phi_i'^2 - \phi_i^2$$

$$-2\phi_{i-1}(\phi_i' - \phi_i) + \phi_i'^2 - \phi_i^2$$

$$+ 2dx^2 \rho_i (\phi_i - \phi_i')\Big\}$$

$$= \frac{1}{2}(\phi_i' - \phi_i)\Big\{ -2\phi_{i+1} + \phi_i' + \phi_i$$

$$-2\phi_{i-1} + \phi_i' - \phi_i - 2dx^2\rho_i \Big\}$$

$$= 2\,\frac{1}{2}(\phi_i' - \phi_i)\Big\{ \phi_i' + \phi_i \underbrace{- \phi_{i+1} - \phi_{i-1} - dx^2\rho_i}_{-2\phi_i'} \Big\}$$

$$= -(\phi_i' - \phi_i)^2 < 0$$

Q: Why the minus sign?

Have you ever seen funny minus sign between different energy terms before!?

Classical Mechanics	Here
Lagrangian	
$\mathcal{L} = T - V$	$\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - \rho(x)\,\phi(x)$
	$x \leftrightarrow \phi$ $t \leftrightarrow x$
Lagra[illegible]	$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial (d\phi/dx)} = 0$
	$-\rho - \frac{d}{dx}\frac{d\phi}{dx} = 0$
$m\ddot{x} = -\frac{dV}{dx} = F$	$d^2\phi/dx^2 = -\rho$
Minimizes action $S = \int \mathcal{L}\, dt$	Poisson Eqn

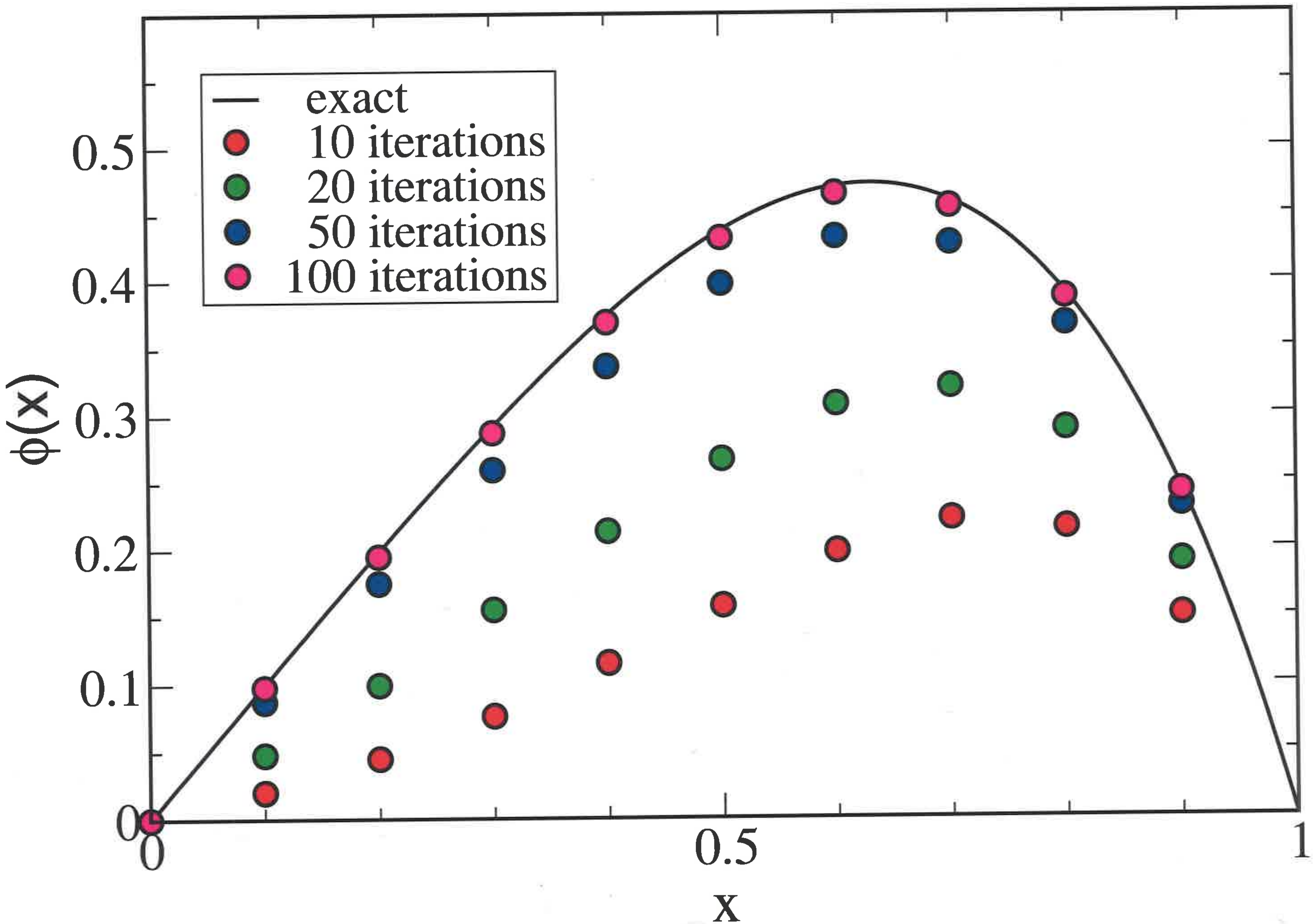
LAST TIME

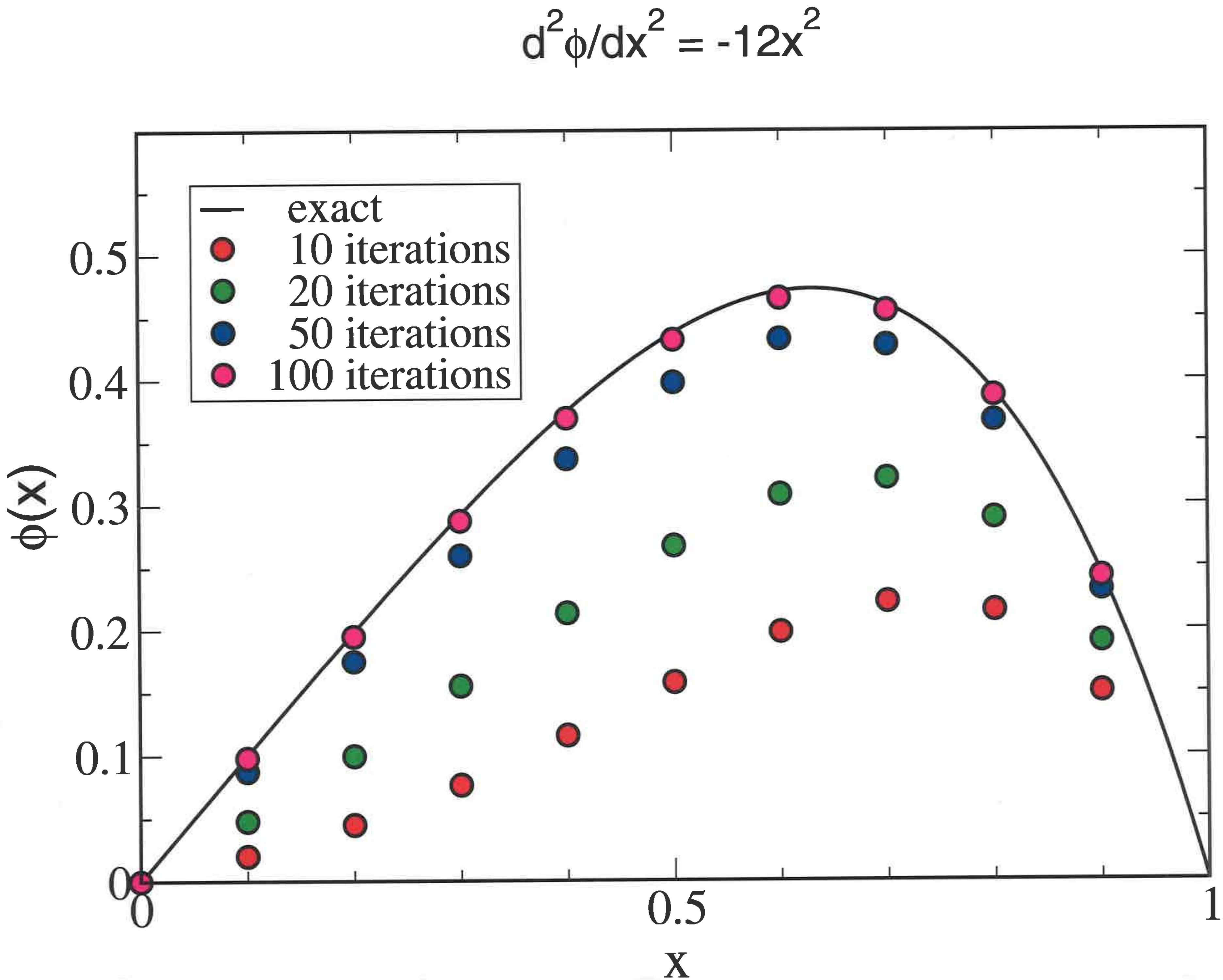
A bit on vectors

Maxwell Eqns

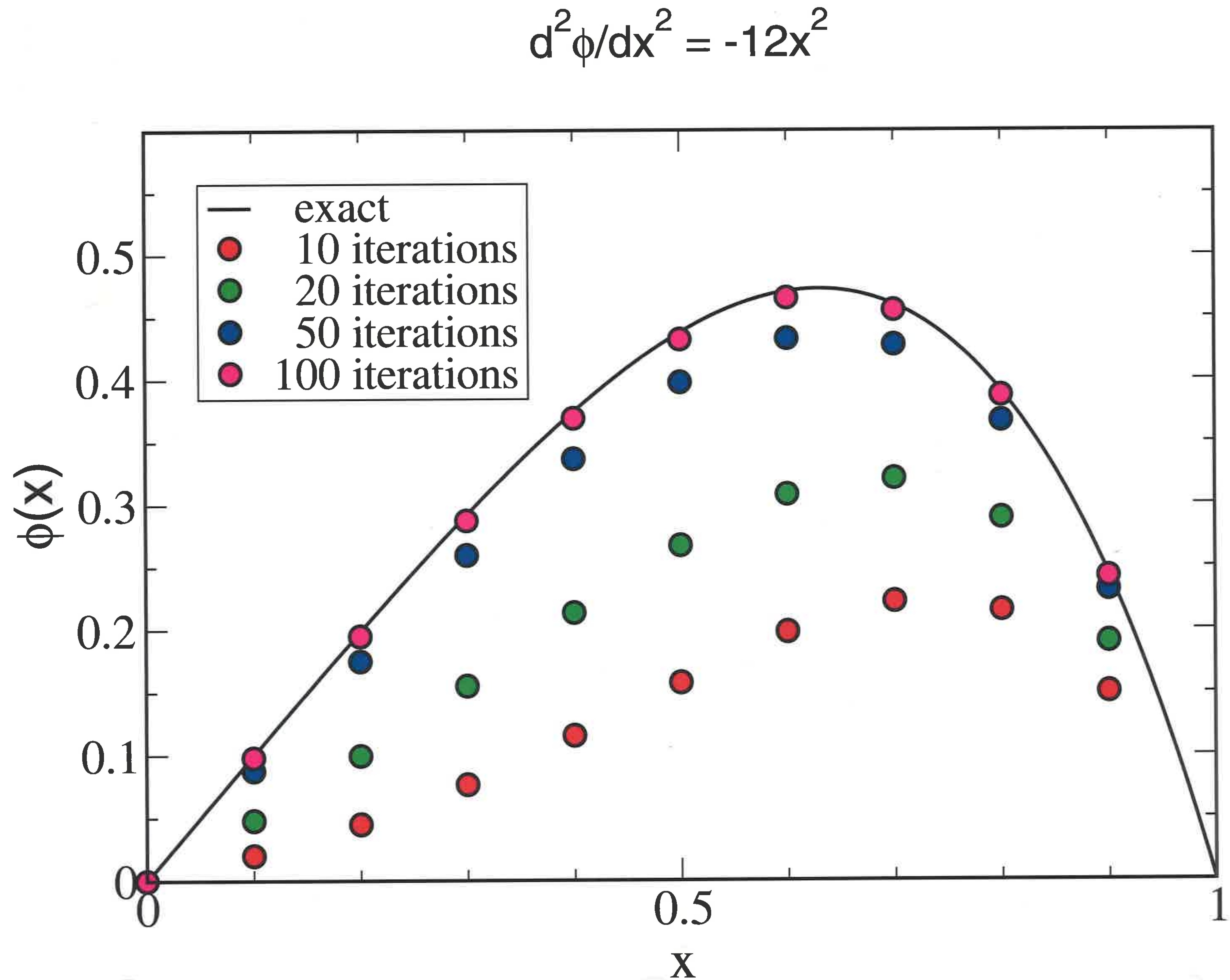
Coulombs law ✓✓

$d^2\phi/dx^2 = -12x^2$



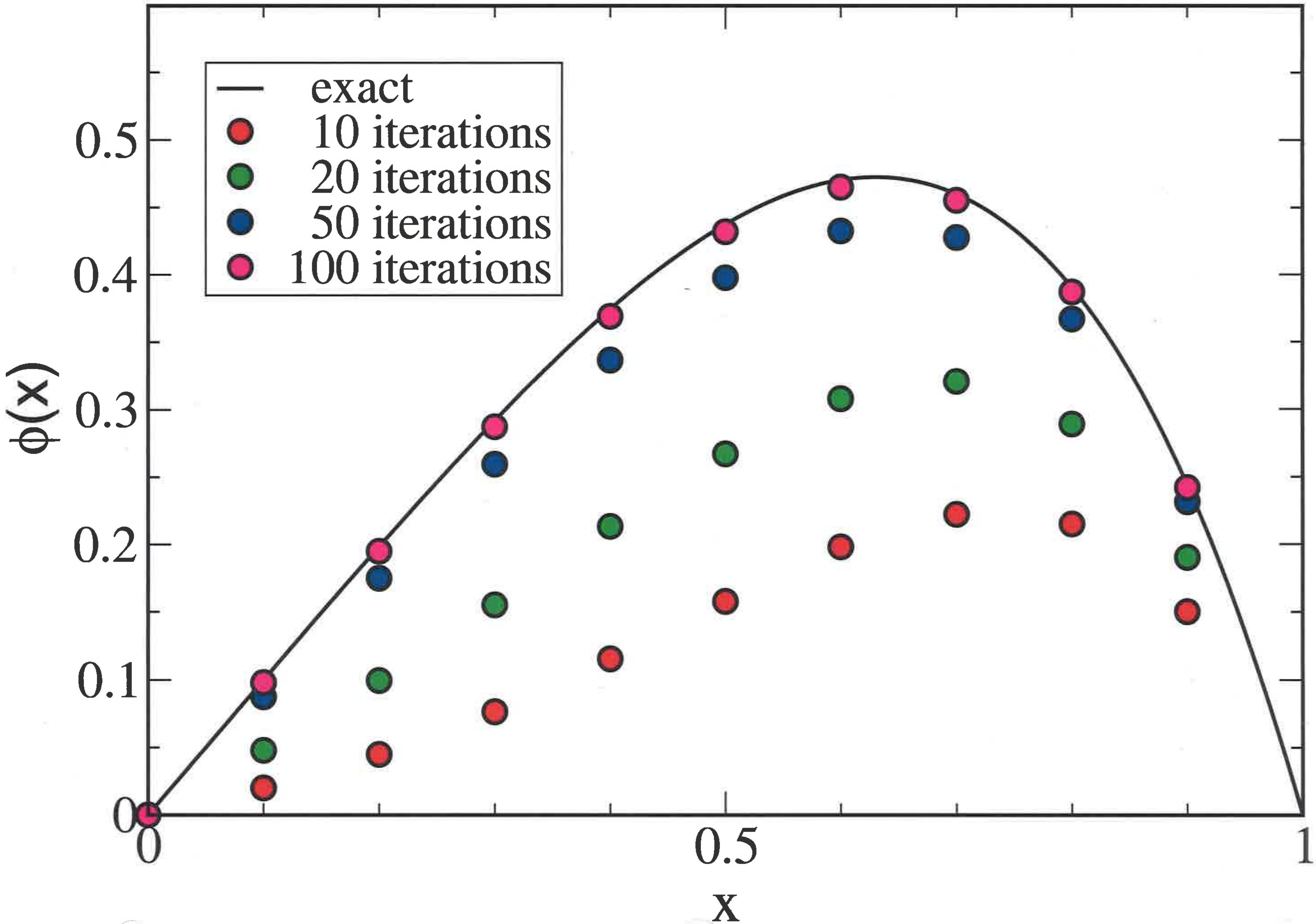


d²φ/dx² = -12x²
exact
10 iterations
20 iterations
50 iterations
100 iterations
φ(x)
0
0.1
0.2
0.3
0.4
0.5
0
0.5
1
x



d²φ/dx² = -12x²
exact
10 iterations
20 iterations
50 iterations
100 iterations
φ(x)
0
0.1
0.2
0.3
0.4
0.5
0
0.5
1
x

$$d^2\phi/dx^2 = -12x^2$$



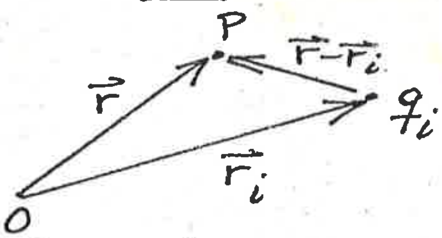
BECAUSE $\vec{E}(\vec{r})$ IS A vector FIELD, IT CAN BE CUMBERSOME TO WORK WITH. HOWEVER, THE SAME INFORMATION MAY BE EMBODIED IN A scalar FUNCTION $\varphi(\vec{r})$, CALLED THE scalar potential (or electric potential). THIS FOLLOWS FROM THE PROPERTY $\vec{\nabla}\times\vec{E}=0$: ANY IRROTATIONAL VECTOR FIELD CAN BE WRITTEN AS THE GRADIENT OF A SCALAR FIELD. WE WRITE

$$\vec{E} = -\vec{\nabla}\varphi,$$

SO THAT $\vec{\nabla}\times\vec{E} = \vec{\nabla}\times(-\vec{\nabla}\varphi) = -(\vec{\nabla}\times\vec{\nabla}\varphi) = 0.$

FOR A POINT CHARGE AT $\vec{r}_i$, WE ASSERT THAT THE SCALAR POTENTIAL IS

$$\varphi(\vec{r}) = \frac{q_i}{|\vec{r}-\vec{r}_i|}$$



(to within a constant --- there's no physics in the constant, so we let it equal zero --- thus $\varphi(\infty)=0$)

TO CHECK THIS, THE ASSOCIATED ELECTRIC FIELD IS

$$\vec{E}(\vec{r}) = -\vec{\nabla}\varphi(\vec{r}) = -q_i\,\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}_i|}\right)$$

$$= \frac{q_i}{|\vec{r}-\vec{r}_i|^2}\,\vec{\nabla}\left(|\vec{r}-\vec{r}_i|\right)$$

$$= \frac{q_i(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3},$$

WHICH IS JUST COULOMB'S LAW AGAIN.

FOR AN ASSEMBLY OF CHARGES, SUPERPOSITION TELLS US THAT THE STATIC POTENTIAL IS

$$\varphi(\vec{r}) = \sum_i \frac{q_i}{|\vec{r}-\vec{r}_i|} \to \sum_i \Delta V_i \frac{\rho(\vec{r}_i)}{|\vec{r}-\vec{r}_i|}$$

$$\to \int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

THUS, IF WE KNOW ρ THROUGH ALL SPACE, WE MAY DETERMINE φ (AND HENCE $\vec{E}$) EVERYWHERE.

WE HAVE AN EQUATION FOR $\varphi(\vec{r})$ IN INTEGRAL FORM: CAN WE GET A DIFFERENTIAL EQUATION? COMBINING $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ AND $\vec{E} = -\vec{\nabla}\varphi$, WE GET

$$\vec{\nabla}\cdot\vec{\nabla}\varphi = \nabla^2\varphi = -4\pi\rho \quad \text{[Poisson's equation]}$$

SOME CONSEQUENCES OF POISSON'S EQUATION:

(i) AT A POINT WHERE $\rho = 0$, φ SATISFIES LAPLACE'S EQUATION $\nabla^2\varphi = 0$.

(ii) AS A CONSEQUENCE, THERE CAN BE NO LOCAL MAXIMUM OR MINIMUM OF φ AT A POINT WHERE $\rho = 0$. IF THERE WAS AN EXTREMUM, THEN $\partial^2\varphi/\partial x_i^2$ WOULD HAVE THE SAME SIGN (+ OR −) FOR $i = 1, 2, 3$ (x-, y-, AND z-COMPONENTS), SO THAT $\nabla^2\varphi = \partial^2\varphi/\partial x^2 + \partial^2\varphi/\partial y^2 + \partial^2\varphi/\partial z^2$ COULD NOT BE ZERO. JUST ANOTHER WAY OF SAYING $\nabla\cdot E = 0$ ACTUALLY VS

(iii) FURTHERMORE, IN A REGION OF SPACE WHERE $\rho = 0$, φ MAY BE SPATIALLY PERIODIC BUT CANNOT BE PERIODIC IN ALL 3 DIMENSIONS. IF φ IS PERIODIC IN THE iTH DIRECTION, THEN $\partial^2\varphi/\partial x_i^2 = -k_i^2\varphi$, WHERE k_i IS THE WAVENUMBER OF THIS PERIODICITY. IF THERE WAS PERIODICITY IN ALL 3 DIMENSIONS, WE WOULD HAVE $\nabla^2\varphi = -(k_x^2 + k_y^2 + k_z^2)\varphi$, WHICH IS NOT COMPATIBLE WITH $\nabla^2\varphi = 0$.

* NEXT PAGE

(iv) A USEFUL MATHEMATICAL RESULT: FROM $\nabla^2\varphi = -4\pi\rho$

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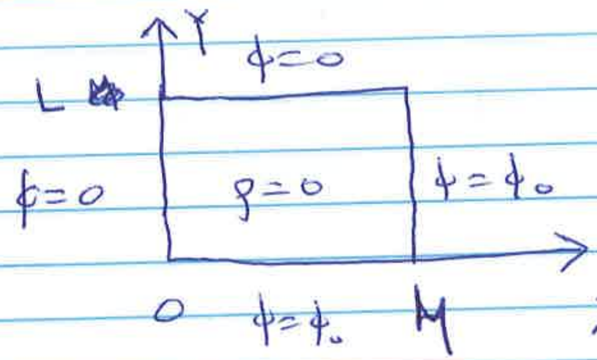
* 1

You have seen an example of this thm before

probably...



Find $\phi(xy)$ in the box

$\boxed{\text{Q: How?}}$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 \phi}{\partial x^2} = -\frac{\partial^2 \phi}{\partial y^2}$$

$$\phi(x,y) = f(x)\, g(y)$$

$$\Rightarrow g \frac{d^2 f}{dx^2} = -f \frac{d^2 g}{dy^2}$$

$$\Rightarrow \frac{1}{f}\frac{d^2 f}{dx^2} = -\frac{1}{g}\frac{d^2 g}{dy^2} = a^2$$

$f(x) = \sinh ax \quad \cosh ax$

$g(y) = \sin ax \quad \cos ax$

} periodic only in $\hat{y}$, not BOTH $\hat{x}, \hat{y}$

$\boxed{\text{Q: WHAT NOW?}}$ BDY COND $\phi(x=0,y) = 0 \Rightarrow$ no $\cosh ax$

$\phi(x, y=0) = 0 \Rightarrow$ no $\cos ay$

$$\phi(xy) = \sinh ax \sin ay$$

$\boxed{\text{Q: WHAT NOW?}}$

*2

$$\phi(x, y=L) = 0 \Rightarrow La = n\pi$$

$$\phi(x,y) = \sinh\frac{n\pi x}{L} \sin\frac{n\pi y}{L}$$

$\boxed{\text{Q What Now?}}$ $\phi(x=M, y) = \phi_0$

superposition $\phi(x,y) = \sum_n b_n \sinh\frac{n\pi x}{L} \sin\frac{n\pi y}{L}$

$$\phi_0 = \phi(M,y) = \sum_n b_n \sinh\frac{n\pi M}{L} \sin\frac{n\pi y}{L}$$

$\int_0^L dy \sin\frac{n'\pi y}{L}$ both sides and use

$$\int_0^L dy \sin\frac{n'\pi y}{L} \sin\frac{n\pi y}{L} = \delta_{nn'}\frac{L}{2}$$

$$\phi_0 \int_0^L dy \sin\frac{n'\pi y}{L} = b_{n'} \sinh\frac{n'\pi M}{L} \frac{L}{2}$$

$$\downarrow$$

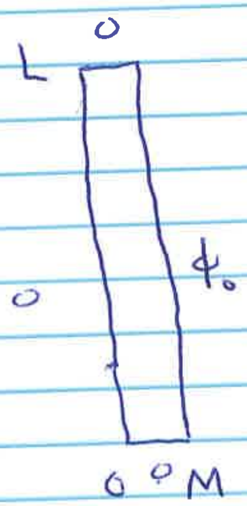
$$-\frac{L}{n'\pi} \cos\frac{n'\pi y}{L}\Big|_0^L = \frac{L}{n'\pi}\left[\cos(n'\pi) - 1\right]$$

$\uparrow$
-1 for n' odd
$+1$ for n' even

$$b_{n'} = \frac{2\phi_0}{L} \frac{1}{\sinh\frac{n'\pi M}{L}} \frac{2L}{n'\pi} \quad n' \text{ odd } \underline{\underline{\text{only}}}$$

*3

$$\phi(x y) = \sum_{n'=1,3,5,\ldots} \frac{4\phi_0}{\pi} \frac{1}{n' \sinh \frac{n'\pi M}{L}} \sinh \frac{n'\pi x}{L} \sin \frac{n'\pi y}{L}$$



Does this make sense?

$\frac{M}{L}$ small $\Rightarrow$ tall skinny box

small # in denom

$\Rightarrow$ big factor $\left(\sinh \frac{n'\pi M}{L}\right)^{-1}$

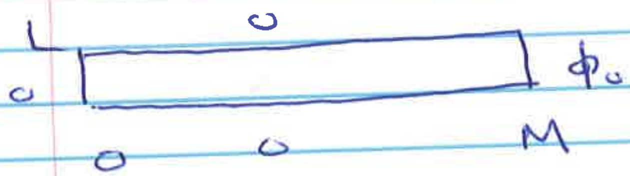
$\phi(x,y)$ ~~must rise f~~ is strongly affected by ϕ_0 throughout box

whereas M/L large $\Rightarrow$ wide flat box

large # in denom

$\Rightarrow$ small factor $\left(\sinh \frac{n'\pi M}{L}\right)$

$\phi(x,y)$ is only weakly influenced away from ϕ zero by ϕ_0



POINT CHARGE AT THE ORIGIN, WE HAVE

$$\nabla^2\left(\frac{q}{r}\right) = -4\pi q\,\delta(\vec{r})$$

AND SO

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta(\vec{r}).$$

WE INTRODUCED THE SCALAR POTENTIAL φ AS A CALCULATIONAL AID – IT'S EASIER TO CALCULATE φ FROM ρ THAN TO CALCULATE $\vec{E}$ DIRECTLY – BUT DOES φ HAS ANY *direct* PHYSICAL SIGNIFICANCE? RECALL THAT $\vec{E}$ HAD THE INTERPRETATION OF THE *force per unit charge* ON A CHARGE q AT POSITION $\vec{r}$ DUE TO OTHER CHARGES:

$$\vec{F}_{elec}(\vec{r}) = q\vec{E}(\vec{r})$$

WE WILL SHOW THAT $\varphi(\vec{r})$ MAY SIMILARLY BE INTERPRETED AS THE *electrostatic potential energy per unit charge* OF A CHARGE q AT POSITION $\vec{r}$ DUE TO OTHER CHARGES. TO SHOW THIS, WE FIRST NOTE THAT FROM NEWTON'S THIRD LAW, IN ORDER TO <u>HOLD</u> A CHARGE q AT POSITION $\vec{r}$ WHERE THE ELECTRIC FIELD IS $\vec{E}(\vec{r})$ WE MUST EXERT THE EXTERNAL FORCE

$$\vec{F}_{ext} = -\vec{F}_{elec} = -q\vec{E}.$$

TO <u>MOVE</u> THE CHARGE AN INFINITESIMAL DISTANCE $d\vec{l}$, WE DO AN AMOUNT OF WORK

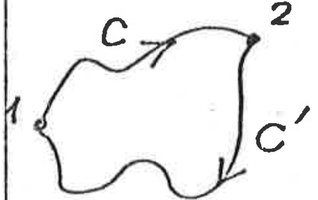
$$dW_{ext} = \vec{F}_{ext}\cdot d\vec{l} = -q\vec{E}\cdot d\vec{l}$$

[WORK DONE BY THE *external agent* AGAINST THE ELECTRIC FIELD]

(This motion is assumed to be infinitely slow so that it's still a statics problem!)

THE NET

$$W_{1\to 2} = \int_1^2 dW_{ext} = -q\int_1^2 \vec{E}\cdot d\vec{l}.$$



SUPPOSE THE MOTION FROM 1 TO 2 FOLLOWED THE PATH C. IF WE NOW RETURN THE CHARGE TO POINT 1 VIA A DIFFERENT PATH C′, HOW MUCH total WORK HAVE WE DONE?

$$W_{1\to 2\to 1} = W^{C}_{1\to 2} + W^{C'}_{2\to 1}$$

$$= \int_{1_C}^{2} dW_{ext} + \int_{2_{C'}}^{1} dW_{ext}$$

$$= -q\left[\int_{1_C}^{2} d\vec{l}\cdot\vec{E} + \int_{2_{C'}}^{1} d\vec{l}\cdot\vec{E}\right]$$

$$= -q\oint_{C+C'} d\vec{l}\cdot\vec{E}$$

$$= -q\int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{E})$$ BY STOKES′ THEOREM

(here S is the area bounded by $C+C'$).

BUT $\vec{\nabla}\times\vec{E}=0$ IN ELECTROSTATICS, SO

$$W_{1\to 2\to 1} = 0 \quad \text{AND} \quad W^{C}_{1\to 2} = -W^{C'}_{2\to 1}$$
$$= W^{C'}_{1\to 2}.$$

THUS THE WORK DONE BY AN EXTERNAL AGENT IN MOVING THE CHARGE FROM ONE POINT TO ANOTHER IS path-independent AND THE ELECTROSTATIC FORCE IS conservative (... $\vec{\nabla}\times\vec{E}=0$

ELECTROSTATIC FORCE IS CENTRAL). WE MAY THEREFORE WRITE $W_{1\to 2}$ AS A FUNCTION OF THE ENDPOINTS ONLY IN TERMS OF THE electrostatic potential energy of the charge q:

P.E. AT $\vec{r}_2$

$$W_{1\to 2} = W(\vec{r}_2) - W(\vec{r}_1) = -q\int_1^2 d\vec{l}\cdot\vec{E}.$$

work done by external agent

P.E. at $\vec{r}_1$

WE NOW define A FUNCTION $\varphi(\vec{r})$ AS

$$\varphi(\vec{r}) = W(\vec{r})/q = \text{p.e. per unit charge at position } \vec{r}.$$

IS THIS THE SAME $\varphi(\vec{r})$ AS BEFORE? WE NOW HAVE

$$\varphi(\vec{r}_2) - \varphi(\vec{r}_1) = -\int_1^2 d\vec{l}\cdot\vec{E}$$

$$= \int_1^2 d\varphi = \int_1^2 d\vec{l}\cdot\vec{\nabla}\varphi,$$

SO THAT WE IDENTIFY

$$\vec{E}(\vec{r}) = -\vec{\nabla}\varphi(\vec{r}),$$

JUST AS BEFORE. SO THE POTENTIAL $\varphi(\vec{r})$ CAN BE REGARDED EITHER AS A MATHEMATICAL ARTIFACT OR AS THE POTENTIAL ENERGY PER UNIT CHARGE AT $\vec{r}$ (relative to some reference point – the only physical quantity of interest is <u>differences</u> in potential energy).

WE HAVE THAT THE WORK DONE BY AN EXTERNAL AGENT TO MOVE A single CHARGE q FROM $\vec{r}_1$ TO $\vec{r}_2$ IS

$$W_{1\to 2} = q[\varphi(\vec{r}_2) - \varphi(\vec{r}_1)].$$

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$\vec{E} = -\nabla\phi$

It may seem "obvious" that $\phi(\vec{x})$ has a physical interpretation, but what about $\vec{A}(\vec{r})$ which we will similarly introduce $\vec{B} = \vec{\nabla} \times \vec{A}$ the vector potential.

Is it a mathematical artifact, or can it be measured somehow?!

Aharonov – Bohm Effect!

AS AN EXTENSION OF THIS, HOW MUCH WORK MUST BE DONE TO ASSEMBLE A GIVEN ARRANGEMENT OF CHARGES q_i, ASSUMING THEY BEGIN AT INFINITY? THIS IS

$$W_N = W_{q_1} + W_{q_2} + \cdots + W_{q_N}$$

work to assemble N charges — work to bring in 1st charge — work to bring in 2nd charge — work to bring in Nth charge

1st charge: $W_{q_1} = 0$ [no other charges present!]

2nd charge: CHARGE q_1 IS AT $\vec{r} = \vec{r}_1$, SO $\varphi(\vec{r}) = \dfrac{q_1}{|\vec{r}-\vec{r}_1|}$.

THUS

$$W_{q_2} = q_2\left[\varphi(\vec{r}_2) - \underset{0}{\underset{\shortparallel}{\varphi(\infty)}}\right]$$

$$= \frac{q_1 q_2}{|\vec{r}_2 - \vec{r}_1|}$$ [note $1 \leftrightarrow 2$ symmetry: it doesn't matter whether 1 or 2 is brought in 1st]

3rd charge: CHARGE q_1 IS AT $\vec{r} = \vec{r}_1$
CHARGE q_2 IS AT $\vec{r} = \vec{r}_2$

SO $$\varphi(\vec{r}) = \frac{q_1}{|\vec{r}-\vec{r}_1|} + \frac{q_2}{|\vec{r}-\vec{r}_2|}$$

AND

$$W_{q_3} = q_3\left[\varphi(\vec{r}_3) - \underset{0}{\underset{\shortparallel}{\varphi(\infty)}}\right]$$

$$= \frac{q_1 q_3}{|\vec{r}_3 - \vec{r}_1|} + \frac{q_2 q_3}{|\vec{r}_3 - \vec{r}_2|}.$$

THUS THE TOTAL WORK DONE TO ASSEMBLE 3 CHARGES IS

$W_3 = W_{q_1} + W_{q_2} + W_{q_3} =$ $q_1 q_2$ + $q_1 q_3$ + $q_2 q_3$ ($1 \leftrightarrow 2 \leftrightarrow$

BY EXTENSION, WE SEE THAT THE WORK DONE TO BRING IN THE iTH CHARGE, ONCE $i-1$ CHARGES ARE ALREADY THERE, IS

$$W_{q_i} = \sum_{j<i} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

AND THE total WORK TO ASSEMBLE N CHARGES IS

$$W_N = \sum_{i=1}^{N} W_{q_i}$$

$$= \sum_{i=1}^{N} \sum_{j<i} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

$$= \frac{1}{2} \sum_{\substack{i,j=1 \\ (i \neq j)}}^{N} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} \quad \left(\text{as } \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{q_j q_i}{|\vec{r}_j - \vec{r}_i|} \right).$$

exclude self-energies of charges

WE MAY REGARD THIS AS THE TOTAL ELECTROSTATIC ENERGY U ASSOCIATED WITH THE ARRANGEMENT OF CHARGES.

FOR A CONTINUOUS DISTRIBUTION OF CHARGE, THIS BECOMES

$$U = \frac{1}{2} \int d^3r \int d^3r' \; \frac{\rho(\vec{r}) \rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$= \frac{1}{2} \int d^3r \, \rho(\vec{r}) \int \frac{d^3r' \, \rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$= \frac{1}{2} \int d^3r \, \rho(\vec{r}) \, \varphi(\vec{r}).$$

THESE FORMS EMPHASIZE THE ELECTROSTATIC ENERGY IN TERMS OF THE CHARGES THAT GENERATE THE FIELDS. WE MAY ALSO REWRITE U IN TERMS OF THE FIELDS WITHOUT EXPLICIT REFERENCE TO THE CHARGES. SINCE

$\rho(\vec{r}) = -\frac{1}{4\pi}\nabla^2\varphi(\vec{r})$, WE CAN WRITE $\qquad \rho(r) = -\epsilon_0 \nabla^2\varphi$

$$U = \frac{1}{2}\int d^3r \left(-\frac{1}{4\pi}\nabla^2\varphi(\vec{r})\right)\varphi(\vec{r})$$

$$= -\frac{1}{8\pi}\int d^3r\ \varphi(\vec{r})\ \vec{\nabla}\cdot\vec{\nabla}\varphi(\vec{r})$$

$$= -\frac{1}{8\pi}\int d^3r\left[\vec{\nabla}\cdot(\varphi(\vec{r})\vec{\nabla}\varphi(\vec{r})) - \vec{\nabla}\varphi(\vec{r})\cdot\vec{\nabla}\varphi(\vec{r})\right]$$

$$= \frac{1}{8\pi}\int d^3r\ |\vec{\nabla}\varphi(\vec{r})|^2 - \frac{1}{8\pi}\int d^3r\ \vec{\nabla}\cdot(\varphi(\vec{r})\vec{\nabla}\varphi(\vec{r})).$$

THE SECOND TERM IS

$$-\frac{1}{8\pi}\int d^3r\ \vec{\nabla}\cdot(\varphi(\vec{r})\vec{\nabla}\varphi(\vec{r}))$$

$$= -\frac{1}{8\pi}\int_S d\vec{S}\cdot\varphi(\vec{r})\vec{\nabla}\varphi(\vec{r})$$ FROM GREEN'S THEOREM;

HERE S IS THE SURFACE BOUNDING THE VOLUME OF INTEGRATION

AS THE VOLUME GROWS TO INCLUDE ALL SPACE, THEN

$\varphi \sim Q/R$ ON S (Q = total enclosed charge)

$\vec{\nabla}\varphi \sim \hat{r}\, Q/R^2$ ON S

SO $$\int_S d\vec{S}\cdot\varphi(\vec{r})\vec{\nabla}\varphi(\vec{r})$$

$$\to \int d\Omega\cdot R^2\cdot\frac{Q^2}{R^3} \propto \frac{1}{R}$$

$$\to 0 \text{ AS } R\to\infty.$$

SO AS THE VOLUME EXPANDS TO FILL ALL SPACE, THIS TERM

$$U = \frac{1}{8\pi}\int d^3r\, |\vec{\nabla}\varphi|^2 = \frac{1}{8\pi}\int d^3r\, E^2(\vec{r}),$$

WHERE THE INTEGRAL IS OVER ALL SPACE. THUS WE MAY REGARD $|E|^2/8\pi$ AS THE ENERGY DENSITY OF THE ELECTROSTATIC FIELD.

WE WILL NOW LOOK FOR SOLUTIONS OF POISSON'S EQUATION $\nabla^2\varphi = -4\pi\rho$ FOR A VARIETY OF CHARGE DISTRIBUTIONS ρ. WE WILL CONSIDER A VARIETY OF METHODS:

(i) DIRECT SOLUTION;
(ii) GREEN'S FUNCTION METHOD AND METHOD OF IMAGES;
(iii) SERIES SOLUTION.

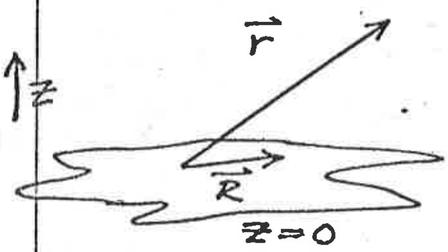
AS AN EXAMPLE OF THE DIRECT SOLUTION, LET'S CONSIDER A FORM OF CHARGE DISTRIBUTION A BIT DIFFERENT FROM WHAT WE'VE SEEN SO FAR. SO FAR WE'VE CONSIDERED ASSEMBLIES OF POINT CHARGES AND CONTINUOUS VOLUME CHARGE DISTRIBUTIONS. THERE ARE ALSO surface charge distributions (2-D) AND line charge distributions (1-D). LET'S CONSIDER A SURFACE CHARGE DISTRIBUTION CONFINED TO THE PLANE $z=0$:

$$\rho(\vec{r}) = \sigma(x,y)\,\delta(z)$$

WHERE $\sigma(x,y)$ IS THE surface charge density. THE TOTAL CHARGE OF THIS FIXED DISTRIBUTION IS THEN

$$Q = \int d^3r\, \rho(\vec{r}) = \int dx \int dy \int dz\, \sigma(x,y)\,\delta(z)$$

$$= \int dx \int dy\, \sigma(x,y).$$



TO FIND THE ELECTROSTATIC POTENTIAL $\varphi(\vec{r})$ DUE TO THIS DISTRIBUTION, WE FIRST WRITE

$$\vec{r} = (x, y, z) = (\vec{R}, z),$$

WHERE $\vec{R} = (x, y)$.

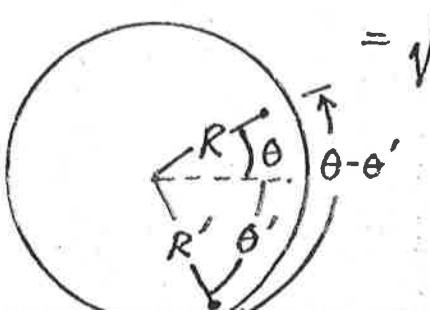
THEN
$$\varphi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$
$$= \int d^2R' \int dz' \; \frac{\sigma(\vec{R}')\,\delta(z')}{|\vec{r}-\vec{r}\,'|}$$
$$= \int d^2R' \; \frac{\sigma(\vec{R}')}{|\vec{r}-\vec{r}\,'|_{z'=0}} ,$$
SINCE $\delta(z') = 0$ FOR $z' \neq 0$.

AS A SPECIFIC EXAMPLE, LET'S CONSIDER A UNIFORMLY CHARGED CIRCULAR DISC. THEN

$$\sigma(R) = \begin{cases} \sigma, & R \leq a \\ 0, & R > a \end{cases}$$

WE MAY WRITE THE DENOMINATOR ABOVE AS

$$|\vec{r}-\vec{r}\,'|_{z'=0} = \sqrt{(\vec{r}-\vec{r}\,')^2_{z'=0}}$$
$$= \sqrt{(\vec{R}-\vec{R}')^2 + z^2} \quad \text{AS } z'=0$$
$$= \sqrt{R^2 + R'^2 - 2RR'\cos(\theta-\theta') + z^2}$$



WHERE WE HAVE USED POLAR COORDINATES TO SPECIFY THE INTEGRATION POINT $(R', \theta', z'=0)$

THEN THE POTENTIAL AT $\vec{r}$ IS

$$\varphi(\vec{r}) = \varphi(R,\theta,z) = \sigma \int_0^a R'dR' \int_0^{2\pi} d\theta' \frac{1}{\sqrt{R^2+R'^2-2RR'\cos(\theta-\theta')+z^2}}$$

THIS IS A DIFFICULT INTEGRAL — INDEED, EVEN THIS RATHER SIMPLE CHARGE DISTRIBUTION GIVES A RATHER COMPLEX EXPRESSION FOR $\varphi(\vec{r})$, WHICH POINTS OUT THE REAL LIMITATIONS OF THE DIRECT METHOD FOR FINDING EXACT SOLUTIONS. WE CAN USE THE SOLUTION IN CERTAIN CASES, THOUGH: IF WE CONFINE OURSELF TO $R=0$ (AN OBSERVATION POINT JUST OVER THE CENTER OF THE DISC OF CHARGE), THEN

$$\varphi(R=0,\theta,z) = \sigma \int_0^a R'dR' \int_0^{2\pi} d\theta' \frac{1}{\sqrt{R'^2+z^2}}$$

$$= 2\pi\sigma \int_0^a dR' \frac{R'}{\sqrt{R'^2+z^2}}$$

(note that this is independent of the sign of z... the field is <u>even</u> in z)

$$= 2\pi\sigma \sqrt{R'^2+z^2}\,\Big|_{R'=0}^{R'=a}$$

$$= 2\pi\sigma\left[\sqrt{a^2+z^2} - \sqrt{z^2}\right]$$

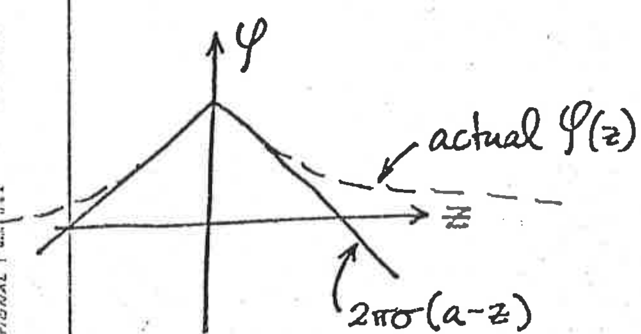
$$= 2\pi\sigma\left[\sqrt{a^2+z^2} - |z|\right]$$

$$= \begin{cases} 2\pi\sigma\left[\sqrt{a^2+z^2} - z\right], & z>0 \\ 2\pi\sigma\left[\sqrt{a^2+z^2} + z\right], & z<0. \end{cases}$$

CONSIDER TWO INTERESTING LIMITING CASES: WHERE THE OBSERVATION POINT IS VERY CLOSE ($|z| \ll a$) OR VERY DISTANT ($|z| \gg a$). FOR $|z| \ll a$, WE TAYLOR EXPAND IN $|z|/a$:

$$\varphi(R=0, z) \cong 2\pi\sigma\left[a\left(1+\frac{z^2}{2a^2}\right)-|z|\right]$$

$$\longrightarrow 2\pi\sigma\left[a-|z|\right].$$ AS z^2/a^2 IS NEGLIGIBLE.



THE ELECTRIC FIELD IS THEN

$$E_z = -\frac{d\varphi}{dz} = \begin{cases} 2\pi\sigma, & z>0 \\ -2\pi\sigma, & z<0 \end{cases}$$

THIS IS THE SAME RESULT AS FOR AN infinite CHARGED PLANE ... THE OBSERVATION POINT IS SUFFICIENTLY CLOSE IN THAT YOU DON'T "SEE" THE EDGES.

AT LARGE DISTANCES, HOWEVER, WHERE $|z| \gg a$, WE HAVE

$$\varphi(R=0, z) \cong 2\pi\sigma\left[|z|\left(1+\frac{a^2}{2z^2}\right)-|z|\right] \quad \left(\text{expanding in } \frac{a}{|z|}\right)$$

$$= \frac{\pi\sigma a^2 |z|}{z^2} = \frac{\pi\sigma a^2}{|z|}.$$

RECALL, HOWEVER, THAT THE TOTAL CHARGE ON THE DISC IS

$$Q = \int d^2R'\,\sigma(\vec{R}') = \int_0^a R'\,dR' \int_0^{2\pi} d\theta' \cdot \sigma$$

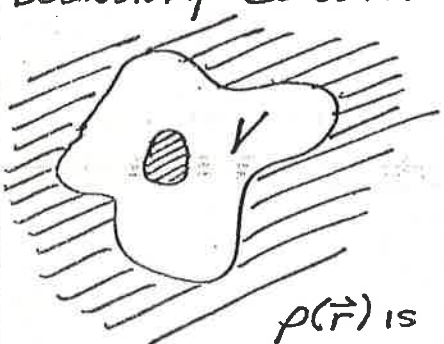
$$= \pi\sigma a^2, \text{ SO THAT}$$

$$\varphi(R=0, z) = \frac{Q}{|z|} \quad \text{FOR } |z| \gg a.$$

THIS IS THE SAME RESULT AS FOR A point charge Q: AT LARGE OBSERVATION DISTANCES, THE INTERNAL STRUCTURE OF THE DISC CAN'T BE SEEN. THE POTENTIAL

OUR HAVING USED THE EXPRESSION $\varphi(\vec{r}) = \int d^3r' \rho(\vec{r}')/|\vec{r}-\vec{r}'|$, WHICH HAS BUILT INTO IT THE CONVENTION THAT $\varphi \to 0$ AS $|\vec{r}| \to 0$ (this is our choice of the arbitrary additive constant for φ).

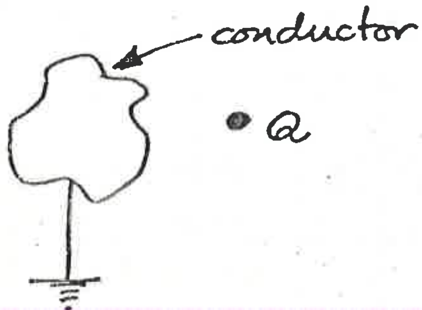
THE ABOVE EXAMPLE (as indeed all of the examples we've seen so far) WORKED WITH THE ASSUMPTION THAT THE CHARGE DISTRIBUTION $\rho(\vec{r})$ WAS KNOWN THROUGHOUT ALL SPACE AND THAT THERE WAS A SIMPLE BOUNDARY CONDITION ON $\varphi(\vec{r})$ AT INFINITY. THERE ARE MANY PHYSICAL SITUATIONS, HOWEVER, WHERE $\rho(\vec{r})$ IS KNOWN THROUGH ONLY A LIMITED VOLUME, BUT THERE ARE ADDITIONAL BOUNDARIES IN THE PROBLEM (which circumscribe the volume) ON WHICH THERE ARE BOUNDARY CONDITIONS ON $\varphi(\vec{r})$. FOR THESE SITUATIONS THE SOLUTION OF POISSON'S EQUATION BECOMES A *boundary-value problem*.



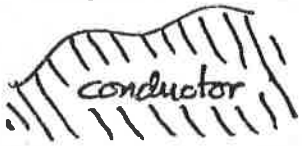
$\rho(\vec{r})$ IS KNOWN IN THE VOLUME V; THERE ARE BOUNDARY CONDITIONS ON $\varphi(\vec{r})$ ON THE SURFACE BOUNDING S.

B.C. on $\varphi(r)$ on the surface
we don't know $\sigma(\vec{r})$ on surface
To use direct method to find $\varphi(r)$ we need not only $\rho(r)$ we bring in but also induced charges on boundaries.

A PARTICULAR, IMPORTANT EXAMPLE OF THIS IS A CHARGE OR CHARGES PLACED IN THE VICINITY OF A *conductor* WHICH IS *grounded* (i.e., CONNECTED BY A WIRE TO INFINITY, WHERE THE POTENTIAL IS ZERO). PHYSICALLY THE CHARGE Q *induces* A CHARGE DISTRIBUTION ON THE CONDUCTOR,



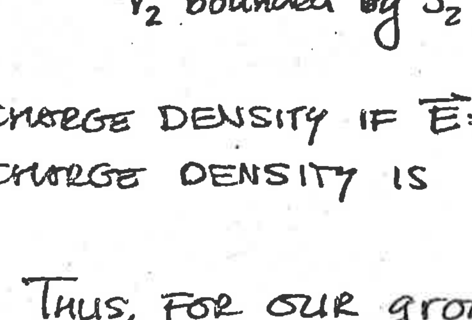
TO SOLVE FOR $\varphi(\vec{r})$ THROUGH ALL SPACE WITH THE DIRECT METHOD WOULD ENTAIL FIRST FINDING THE CHARGE DISTRIBUTION ON THE SURFACE. THE SIMPLER METHOD IS TO EXPLOIT THE Boundary conditions ON $\varphi(\vec{r})$ AT THE SURFACE OF THE CONDUCTOR. TO DERIVE THIS CONDITION, WE NOTE THE FOLLOWING:



(a) $\vec{E}$ MUST BE NORMAL TO THE CONDUCTOR SURFACE (the conductor is assumed to be perfect), SINCE IF THERE WAS A TANGENTIAL COMPONENT OF $\vec{E}$, CHARGES WOULD MOVE ON THE SURFACE UNTIL THE TANGENTIAL COMPONENT VANISHED.

SINCE $\vec{E} = -\vec{\nabla}\varphi$, IT FOLLOWS THAT THE CONDUCTOR'S SURFACE IS AN *equipotential surface* (CONSTANT φ).

(b) $\vec{E} = 0$ INSIDE THE CONDUCTOR, SINCE ANY NONZERO $\vec{E}$ WOULD CAUSE CHARGES TO MOVE IN THE VOLUME UNTIL THE FIELD VANISHED. AGAIN USING $\vec{E} = -\vec{\nabla}\varphi$, WE CONCLUDE THAT THE ENTIRE CONDUCTOR IS AN *equipotential volume*.



APPLYING GAUSS' LAW $\int \vec{E} \cdot d\vec{S} = 4\pi Q$ TO VOLUME 2, WE SEE THAT $Q = 0$: THERE IS NO NET CHARGE WITHIN THE CONDUCTOR AT ANY POINT. APPLYING IT TO VOLUME 1, WE SEE THAT THERE WILL BE A SURFACE CHARGE DENSITY IF $\vec{E} \neq 0$ AT THE SURFACE: THE SURFACE CHARGE DENSITY IS

$\sigma = E_n / 4\pi$, WHERE E_n IS THE outward NORMAL COMPONENT.

THUS, FOR OUR grounded CONDUCTOR, $\varphi = 0$ THROUGHOUT THE CONDUCTOR. TO SOLVE FOR $\varphi(\vec{r})$ OUTSIDE THE CONDUCTOR, WE MUST SOLVE

$$\nabla^2 \varphi(\vec{r}) = -4\pi\rho(\vec{r})$$ WITH BOUNDARY CONDITIONS

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FOR AN *ungrounded* CONDUCTOR, $\varphi(\vec{r}) = \varphi_0$ (some constant) ON THE SURFACE.

How CAN WE SOLVE THIS BOUNDARY-VALUE PROBLEM? THE SOLUTION WE'LL PRESENT — THE METHOD OF *Green's functions* — INVOLVES FORMING A FUNCTION WHICH HAS ALL INFORMATION ABOUT THE BOUNDARY CONDITIONS BUILT IN. SUPPOSE WE HAVE AN INHOMOGENEOUS DIFFERENTIAL EQUATION (LIKE POISSON'S EQUATION) OF THE FORM

$$\mathcal{D}\varphi(\vec{r}) = -4\pi f(\vec{r})$$

; WE TAKE THE B.C. TO BE $\varphi(\vec{r}) = 0$ ON THE BOUNDARIES.

differential operator in $\vec{r}$

known source function

WE SOLVE THIS BY SEARCHING FOR A FUNCTION OF *two* VARIABLES $G(\vec{r},\vec{r}\,')$, CALLED A *Green's function*, WHICH SATISFIES

$$\mathcal{D}G(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,')$$

WITH B.C. $G(\vec{r},\vec{r}\,') = 0$ FOR $\vec{r}$ ON S.

IF WE CAN FIND $G(\vec{r},\vec{r}\,')$, THEN $\varphi(\vec{r})$ IS FOUND BY

$$\varphi(\vec{r}) = \int_V d^3r'\, G(\vec{r},\vec{r}\,')\, f(\vec{r}\,').$$

THIS SATISFIES THE DIFFERENTIAL EQUATION, SINCE

$$\begin{aligned}
\mathcal{D}\varphi(\vec{r}) &= \int_V d^3r'\, \mathcal{D}G(\vec{r},\vec{r}\,')\, f(\vec{r}\,') \\
&= \int d^3r' \left(-4\pi\delta(\vec{r}-\vec{r}\,')\right) f(\vec{r}\,') \\
&= f(\vec{r});
\end{aligned}$$

FURTHERMORE THE B.C. ARE SATISFIED, SINCE

WE SEE THAT $G(\vec{r},\vec{r}\,')$ MEASURES THE AMOUNT OF *influence* THE SOURCE FUNCTION $f(\vec{r})$ HAS ON THE FUNCTION $\varphi(\vec{r})$, SO THAT $G(\vec{r},\vec{r}\,')$ MAY BE THOUGHT OF AS AN *influence function* OR A *response function*. NOTE ALSO THAT $G(\vec{r},\vec{r}\,')$ IS INDEPENDENT OF THE SOURCE FUNCTION $f(\vec{r})$: ONCE $G(\vec{r},\vec{r}\,')$ IS KNOWN FOR A GIVEN SET OF B.C., YOU CAN FIND $\varphi(\vec{r})$ FOR <u>any</u> SOURCE FUNCTION.

WE'VE ALREADY SEEN A GREEN'S FUNCTION FOR THE CASE WHERE THE ONLY SURFACE IS AT INFINITY: FOR POISSON'S EQUATION

$$\nabla^2 \varphi(\vec{r}) = -4\pi\rho(\vec{r}),$$

WITH B.C. $\varphi(\vec{r}) \to 0$ AS $|\vec{r}| \to \infty$,

THE GREEN'S FUNCTION SATISFIES

$$\nabla_r^2 G(\vec{r},\vec{r}\,') = -4\pi\,\delta(\vec{r}-\vec{r}\,'),$$

WITH B.C. $G(\vec{r},\vec{r}\,') = 0$ AS $|\vec{r}| \to \infty$.

BUT RECALL THAT

$$\nabla_r^2 \frac{1}{|\vec{r}-\vec{r}\,'|} = -4\pi\,\delta(\vec{r}-\vec{r}\,'); \quad \ni \quad \frac{1}{|\vec{r}-\vec{r}\,'|} \to 0 \text{ as } |\vec{r}| \to \infty$$

HENCE FOR THIS EXAMPLE $G(\vec{r},\vec{r}\,') = \dfrac{1}{|\vec{r}-\vec{r}\,'|}$

AND

$$\varphi(\vec{r}) = \int d^3r'\, G(\vec{r},\vec{r}\,')\,\rho(\vec{r}\,')$$

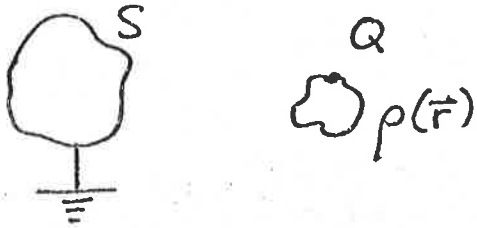
$$= \int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|},$$

JUST AS WE'VE BEEN DOING.

THUS THE "DIRECT METHOD" FOR FINDING $\varphi(\vec{r})$ IN FACT IS TO USE THE GREEN'S FUNCTION $G(\vec{r},\vec{r}\,') = 1/|\vec{r}-\vec{r}\,'|$.

FOR THIS SIMPLEST CASE, $G(\vec{r},\vec{r}\,') = G(\vec{r}\,',\vec{r})$ [G is *symmetric*]; THIS IS IN FACT TRUE IN GENERAL.

WE MAY NOW RETURN TO THE MORE COMPLEX PROBLEM OF A CHARGE NEAR A GROUNDED CONDUCTOR: TO SOLVE FOR $\varphi(\vec{r})$ IN THE VOLUME BOUNDED BY THE CONDUCTOR AND BY INFINITY (i.e. THE VOLUME EXTERNAL TO THE CONDUCTOR), WE HAVE



$$\nabla^2 \varphi(\vec{r}) = -4\pi\rho(\vec{r}),$$

WITH B.C. (i) $\varphi(\vec{r}) = 0$ ON S

(ii) $\varphi(\vec{r}) \to 0$ AS $|\vec{r}| \to \infty$.

THE CORRESPONDING GREEN'S FUNCTION SATISFIES

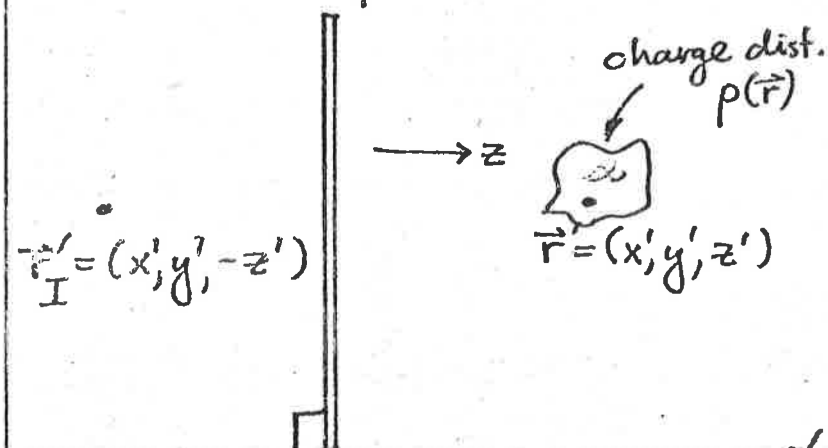
$$\nabla_r^2 G(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,')$$

WITH B.C. (i) $G(\vec{r},\vec{r}\,') = 0$ FOR $\vec{r}$ ON S

(ii) $\varphi(\vec{r},\vec{r}\,') \to 0$ AS $|\vec{r}| \to \infty$.

THE SIMPLE GREEN'S FUNCTION $1/|\vec{r}-\vec{r}\,'|$ SATISFIES (ii) BUT NOT (i) ... SO WE NEED SOMETHING BETTER.

LET US CONSIDER THE SPECIAL CASE OF A GROUNDED CONDUCTING *plane*: WE WISH TO FIND THE FIELD TO THE RIGHT OF THE PLANE.



TO SATISFY (i), WE NOTE THAT WE CAN WRITE

$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{1}{|\vec{r}-\vec{r}_I\,'|}$$

Differential equation:

$$\nabla_r^2 G(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,') + 4\pi\delta(\vec{r}-\vec{r}\,'_I).$$

SINCE $\vec{r}\,'$ WILL BE IN THE RIGHT HALF-SPACE (where the charge distribution is), $\vec{r}\,'_I$ WILL BE IN THE *left* HALF-SPACE. SINCE THE OBSERVATION POINT $\vec{r}$ IS ON THE RIGHT, $\vec{r}$ AND $\vec{r}\,'_I$ NEVER COINCIDE AND THE SECOND δ-FUNCTION VANISHES. So

$$\nabla_r^2 G(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,'), \text{ AS IT SHOULD.}$$

Boundary condition (i):

FOR $z=0$,

$$G(\vec{r},\vec{r}\,') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+z'^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(-z')^2}}$$

$= 0$, AS IT SHOULD.

Boundary condition (ii):

FOR $|\vec{r}|\to\infty$, $G(\vec{r},\vec{r}\,')\to 0$, AS IT SHOULD.

SO WE HAVE THE RIGHT GREEN'S FUNCTION. FOR ANY $\rho(\vec{r})$, THEN, THE POTENTIAL IN THE RIGHT HALF-SPACE (R.H.S.) IS

$$\varphi(\vec{r}) = \int_{RHS} d^3r'\, G(\vec{r},\vec{r}\,')\rho(\vec{r}\,')$$

$$= \int_{RHS} d^3r'\,\rho(\vec{r}\,')/|\vec{r}-\vec{r}\,'| - \int_{RHS} d^3r'\,\rho(\vec{r}\,')/|\vec{r}-\vec{r}\,'_I|$$

$$= \int_{\text{all space}} d^3r' \frac{[\rho(\vec{r}\,') - \rho(\vec{r}\,'_I)]}{|\vec{r}-\vec{r}\,'|}$$

BY LETTING $z' \to -z'$ IN THE 2ND TERM.

If $\rho(\vec{r})=0$ in LHS

THIS IS THE SAME POTENTIAL WE WOULD GET IF THERE WERE NO GROUNDED PLANE BUT INSTEAD THERE WAS A ... REFLECTED ABOUT THE PLANE $z=0$

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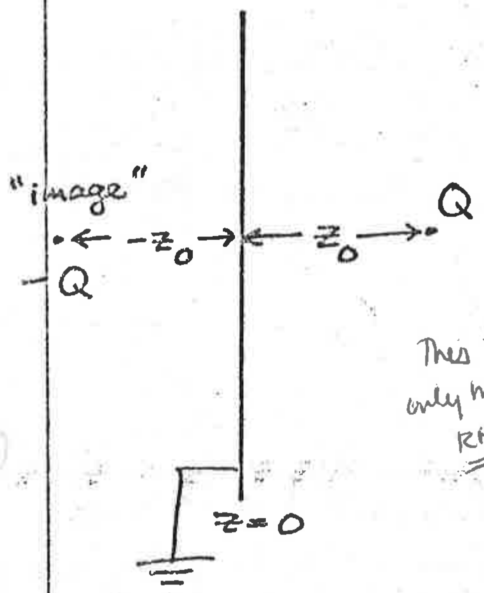
Example: FOR A POINT CHARGE AT $\vec{r}_0 = (0, 0, z_0)$,

$$\rho(\vec{r}\,') = Q\delta(\vec{r}\,' - \vec{r}_0)$$
$$= Q\delta(x')\delta(y')\delta(z' - z_0)$$

SO THAT

$$\varphi(\vec{r}) = \frac{Q}{|\vec{r} - \vec{r}_0|} - \frac{Q}{|\vec{r} - \vec{r}_{0_I}|}$$

I for "image"



THE ELECTRIC FIELD IS

$$\vec{E}(\vec{r}) = -\vec{\nabla}\varphi(\vec{r})$$
$$= \frac{Q(\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^3} - \frac{Q(\vec{r} - \vec{r}_{0_I})}{|\vec{r} - \vec{r}_{0_I}|^3};$$

This $\vec{E}$ is only the $\vec{E}$ on RHS

ON THE SURFACE OF THE CONDUCTING PLANE $(z=0)$, THIS IS

$$\vec{E}(x, y, z=0) = \frac{Q(\hat{i}x + \hat{j}y - \hat{k}z_0)}{(x^2 + y^2 + z_0^2)^{3/2}} - \frac{Q(\hat{i}x + \hat{j}y + \hat{k}z_0)}{(x^2 + y^2 + z_0^2)^{3/2}}$$

$$= -\hat{k} \cdot \frac{2Qz_0}{(x^2 + y^2 + z_0^2)^{3/2}} \quad \frac{1}{4\pi\epsilon_0}$$

WHICH IS NORMAL TO THE SURFACE AS IT MUST BE. THE SURFACE CHARGE DENSITY IS THEN

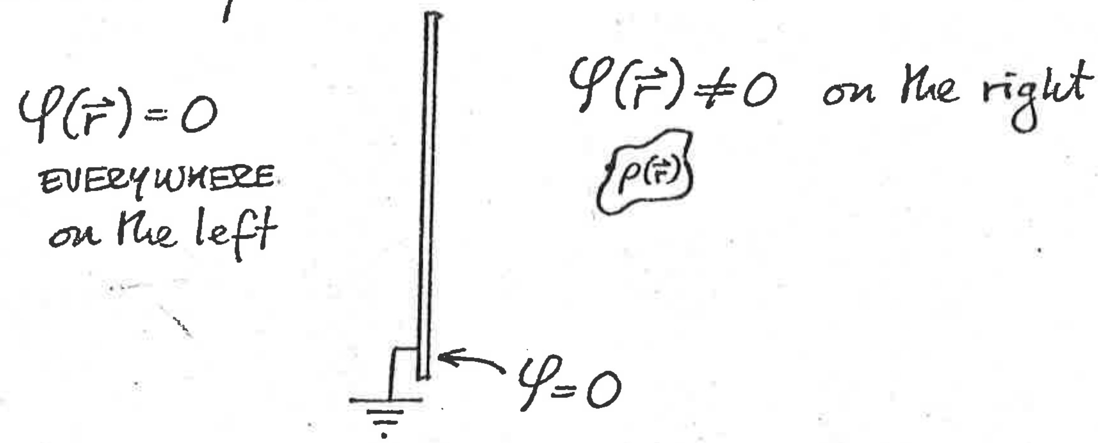
$$\sigma(x, y) = -\frac{2Qz_0}{4\pi(x^2 + y^2 + z_0^2)^{3/2}}, \qquad \sigma = E\epsilon_0$$

AND THE TOTAL CHARGE ON THE RIGHT-HAND CONDUCTOR SURFACE IS

$$Q_{cond} = \int dx \int dy\, \sigma(x, y) = \int_0^{2\pi} d\phi \int_0^{\infty} \rho\, d\rho\, \sigma(\rho, \phi)$$

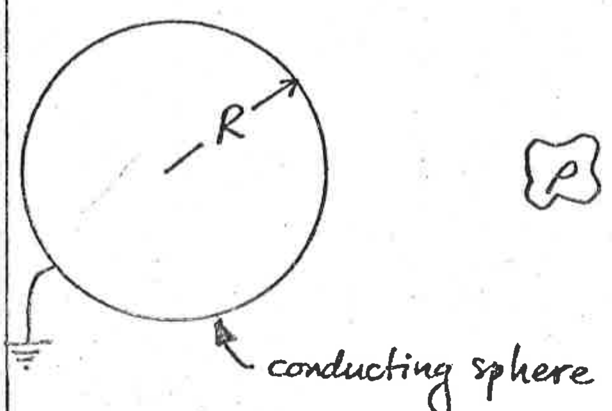
SO A CHARGE $-Q$ IS INDUCED ON THE CONDUCTOR SURFACE independent of the distance z_0, JUST AS THE IMAGE CHARGE IS $-Q$ INDEPENDENT OF THE DISTANCE z_0.

WE NOTE THAT IF $\rho(\vec{r})=0$ IN THE RIGHT HALF-SPACE, $\varphi(\vec{r})=0$ THERE DUE TO THE GREEN'S FUNCTION. THE SAME HOLDS FOR THE LEFT HALF-SPACE: THUS, IF $\rho(\vec{r})\neq 0$ ON THE RIGHT AND $\rho(\vec{r})=0$ ON THE LEFT, WE HAVE



... THE CONDUCTING PLANE ACTS AS A PERFECT "SHIELD". THUS THE IMAGE CHARGE IS LIKE A VIRTUAL IMAGE IN OPTICS: IT'S NOT "really" THERE, BUT THINGS on the RHS look LIKE IT'S THERE.

WE CAN APPLY THE method of images FOR FINDING $G(\vec{r},\vec{r}\,')$ AND THUS $\varphi(\vec{r})$ FOR GROUNDED CONDUCTORS OF DIFFERENT GEOMETRIES. IN PARTICULAR CONSIDER A GROUNDED CONDUCTING sphere OF RADIUS R. WE'LL LOOK FOR AN IMAGE OF THE POINT $\vec{r}\,'$, WHERE $|\vec{r}\,'|>R$, AS $\vec{r}\,'_I$, WHERE $|\vec{r}\,'_I|<R$. WE ARE TRYING IN ALL THIS TO SOLVE THE exterior problem OF THE FIELD $\varphi(\vec{r})$ OUTSIDE THE SPHERE DUE TO A CHARGE DENSITY $\rho(\vec{r})$ OUTSIDE THE SPHERE.

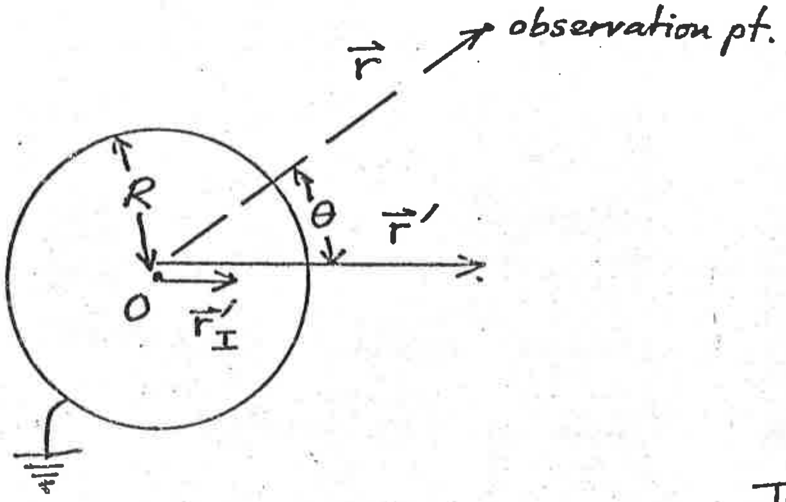


AGAIN, WE REQUIRE $\nabla_r^2 G(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,')$ FOR $|\vec{r}|>R$,

WITH B.C. (i) $G(\vec{r},\vec{r}\,')=0$ FOR $|\vec{r}|=R$

(ii) $G(\vec{r},\vec{r}\,')\to 0$ AS $|\vec{r}|\to\infty$.

USING AN IMAGE AS FOR THE CONDUCTING PLANE, WE TRY



$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} + \frac{\alpha}{|\vec{r}-\vec{r}_I\,'|};$$

BY SYMMETRY WE EXPECT $\vec{r}_I\,'$ TO BE PARALLEL TO $\vec{r}\,'$, WITH $|\vec{r}\,'|>R$ AND $|\vec{r}_I\,'|<R$.

THEN, OUTSIDE THE SPHERE,

$$\nabla_r^2 G(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,') - \underbrace{4\pi\alpha\,\delta(\vec{r}-\vec{r}_I\,')}_{=\,0 \text{ AS } |\vec{r}_I\,'|<R}$$

$$= -4\pi\delta(\vec{r}-\vec{r}\,'), \text{ AS DESIRED.}$$

THE B.C. AS $|\vec{r}|\to\infty$ IS ALSO IMMEDIATELY SATISFIED. TO SATISFY THE B.C. AT $|\vec{r}|=R$, WE HAVE

$$G(\vec{r},\vec{r}\,')\Big|_{|\vec{r}|=R} = \frac{1}{\sqrt{R^2+r'^2-2Rr'\cos\theta}} + \frac{\alpha}{\sqrt{R^2+r_I'^2-2Rr_I'\cos\theta}}$$

$= 0$, INDEPENDENT OF θ; CLEARLY $\alpha<0$.

FOR THIS TO BE SATISFIED, THE TWO TERMS MUST HAVE THE SAME AZIMUTHAL DEPENDENCE. SIMPLIFYING, WE HAVE

$$\frac{R^2+r_I'^2-2Rr_I'\cos\theta}{\ldots} = \alpha^2, \text{ INDEPENDENT OF } \theta,$$

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SO THAT

(a) $R^2 + r_I'^2 = \alpha^2 (R^2 + r'^2)$, AND

(b) $2Rr_I' = 2R\alpha^2 r'$.

FROM (b), WE SEE THAT $\alpha^2 = r_I'/r' < 1$.

USING THIS IN (a) WE GET

$$R^2 + r_I'^2 = \frac{r_I'}{r'}(R^2 + r'^2), \text{ so}$$

$$r_I'^2 r' - r_I'(R^2 + r'^2) + R^2 r' = 0$$

AND SO
$$r_I' = \frac{R^2 + r'^2 \pm \sqrt{(R^2+r'^2)^2 - 4R^2 r'^2}}{2r'}$$

$$= \frac{R^2 + r'^2 \pm \sqrt{(R^2 - r'^2)^2}}{2r'}$$

$$= \begin{cases} R^2/r' = R\, R/r' < R \\ \text{or} \\ r'. \end{cases}$$

$r_I' = r'$ DOESN'T WORK SINCE THEN $|\vec{r}_I'| > R$, SO THAT THE IMAGE WOULD BE OUTSIDE THE SPHERE.

THUS $r_I' = R^2/r' < R$, SINCE $r' > R$

THEN

$$\alpha = -\sqrt{r_I'/r'} = -\sqrt{R^2/r'^2}$$

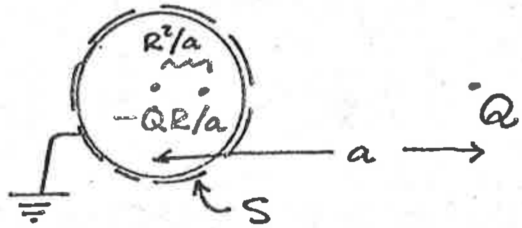
$$= -R/r'. \qquad \text{(recall that } \alpha < 0)$$

So $$G(\vec{r}-\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} - \frac{(R/r')}{|\vec{r}-(R^2/r'^2)\vec{r}'|}$$

FOR A POINT CHARGE AT $|\vec{r}'| = a$, THEN, $\rho(\vec{r}) = Q\,\delta(\vec{r}-\vec{a})$

$$\varphi(\vec{r}) = \int d^3r'\, \rho(\vec{r}')\, G(\vec{r},\vec{r}')$$
$$= \frac{Q}{|\vec{r}-\vec{a}|} - \frac{(QR/a)}{|\vec{r}-(R^2/a^2)\vec{a}|}$$

THE IMAGE CHARGE, AT $|\vec{r}| = R^2/a$, HAS CHARGE $-QR/a$. AS $R \to 0$, THE MODIFICATION OF $\varphi(\vec{r})$ DUE TO THE CONDUCTOR VANISHES. AS $R \to \infty$, $a \to \infty$ AS WELL (the distance from the charge to the radius of curvature of the conductor), AND IT'S EASY TO SEE THAT $R/a \to 1$ SO THAT THE IMAGE CHARGE BECOMES $-Q$, AS FOR THE FLAT CONDUCTING PLATE. WITHOUT CALCULATING $\vec{E}(\vec{r})$, WE CAN STILL CALCULATE THE TOTAL INDUCED CHARGE ON THE SPHERE: FOR THE FIELD OUTSIDE THE SPHERE IS JUST THAT FORMED BY Q (AT $\vec{r}=\vec{a}$) AND THE IMAGE CHARGE $-QR/a$ (AT $\vec{r}=(R^2/a^2)\vec{a}$)



THUS THE TOTAL ENCLOSED CHARGE WITHIN S IS $-QR/a$, AND SO THE TOTAL CHARGE ON THE SPHERE IS $-QR/a$, i.e. THE SAME AS THAT OF THE IMAGE CHARGE.

<u>Question</u>: WHAT IS THE *force* ON THE CHARGE Q AT $\vec{r}=\vec{a}$?

CLEARLY IT'S JUST THE FORCE DUE TO THE IMAGE CHARGE:

$$\vec{F} = \frac{Q(-QR/a)}{|\vec{a}-(R^2/a^2)\vec{a}|^2} = \frac{-Q^2R/a}{a^2(1-R^2/a^2)^2}$$

$$Q^2Ra$$

ONE AMUSING CONCLUSION OF OUR CALCULATIONS WITH IMAGES: *for any two charges* Q_1 *and* Q_2*, the resulting equipotential surface is either a plane* (FOR $Q_1 = \pm Q_2$) *or a sphere* (FOR $Q_1 \neq \pm Q_2$).

THE METHOD OF IMAGES WE'VE USED FOR FINDING THE GREEN'S FUNCTION UNFORTUNATELY SMACKS A BIT OF BLACK MAGIC, AND IT WOULD BE NICE TO HAVE A MORE METHODICAL WAY TO GENERATE $G(\vec{r},\vec{r}\,')$ FOR A SPECIFIC GEOMETRY. WE'LL RETURN TO THIS QUESTION LATER. LET US CONCLUDE OUR PRESENT DISCUSSION OF GREEN'S FUNCTIONS BY NOTING THAT FOR BOUNDARY CONDITIONS OTHER THAN $\varphi(\vec{r}) = 0$ FOR $\vec{r}$ ON THE BOUNDARIES, WE MUST USE

$$\varphi(\vec{r}) = \int_V d^3r' \rho(\vec{r}\,') G(\vec{r},\vec{r}\,') + \frac{1}{4\pi}\int_S ds' \left[G(\vec{r},\vec{r}\,') \frac{\partial \varphi(\vec{r}\,')}{\partial n'} - \varphi(\vec{r}\,') \frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'} \right],$$

we have discussed cases where Φ and G vanish on S' so that we were okay in neglecting the extra terms.

WHERE THE NORMAL DERIVATIVE IS IN THE DIRECTION OUT OF THE VOLUME. IF $\varphi(\vec{r})$ IS SPECIFIED ON S, THEN look for $G(r,r')$ such that $G(\vec{r},\vec{r}\,') = 0$ FOR $\vec{r}$ ON S (*as for our special case where* $\varphi(\vec{r}) = 0$ *on* S); IF $\partial\varphi(\vec{r}\,')/\partial n'$ IS SPECIFIED ON S, THEN look for G such that $\partial G(\vec{r},\vec{r}\,')/\partial n' = -4\pi/S$ FOR $\vec{r}$ ON S, WHERE S IS THE TOTAL AREA OF THE BOUNDARIES. In our problem of sphere too we get $S' = \infty$ so this vanishes. AN EXAMPLE OF USING THIS MORE COMPLETE FORM FOR $\varphi(\vec{r})$ IS GIVEN IN THE HOMEWORK. NOTE THAT *no matter what* $\varphi(\vec{r})$ IS SPECIFIED, THE B.C. FOR $G(\vec{r},\vec{r}\,')$ DON'T CHANGE, SO THAT WE COULD USE OUR $G(\vec{r},\vec{r}\,')$ FOR THE PLANE AND THE SPHERE EVEN IF THEY WERE AT A SPECIFIED *nonzero*