

Bessel Function in Statistical Mechanics!



1-d chain
of magnetic
moments
which can rotate
in a plane

$$E = -K \sum_{\ell=1}^N \cos(\theta_{\ell} - \theta_{\ell+1})$$

ground state is $\theta_{\ell} = \text{const}$ $E = -NJ$

"ferromagnetism"

$$Z = \sum_{\text{states}} e^{-\beta E}$$

$$= \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \dots \int_0^{2\pi} d\theta_N e^{+\beta K \sum_{\ell=1}^N \cos(\theta_{\ell} - \theta_{\ell+1})}$$

$$= \int_0^{2\pi} d\theta_1 e^{\beta K \cos(\theta_1 - \theta_2)} \int_0^{2\pi} d\theta_2 e^{\beta K \cos(\theta_2 - \theta_3)} \dots$$

$$\int_0^{2\pi} d\theta_2 e^{\beta K \cos(\theta_1 - \theta_2)} e^{\beta K \cos(\theta_2 - \theta_3)}$$

reminds us of what

$$M^2(\theta_1, \theta_3) = \int d\theta_2 M(\theta_1, \theta_2) M(\theta_2, \theta_3)$$

$$Z = M \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_N M^{N-1}(\theta_1, \theta_N)$$

PBC $K \cos(\theta_1 - \theta_N)$ term

1d finite dim matrices
+ Hilbert space
Already encountered
 $\langle x | p \rangle = e^{ipx/\hbar}$

$$\langle x | \psi \rangle = \psi(x)$$

$$\langle x | p | \psi \rangle = \frac{\hbar}{i} \frac{\partial \psi}{\partial x}$$

$$\langle x | p | x' \rangle \langle x' | \psi \rangle$$

$$\langle x | p | x' \rangle = \frac{\hbar}{i} \delta(x-x')$$

SM-2A

How:

$$e^{\frac{x}{2}(t - 1/t)} = \sum J_n(x) t^n$$

$$e^{ix \sin \theta} = \sum J_n(x) e^{in\theta}$$

$$\left. \begin{aligned} t &= e^{i\theta} \\ e^{\frac{x}{2} 2\cos \theta} &= \sum I_n(x) e^{in\theta} \\ e^{x \cos \theta} &= \sum I_n(x) e^{in\theta} \end{aligned} \right\}$$

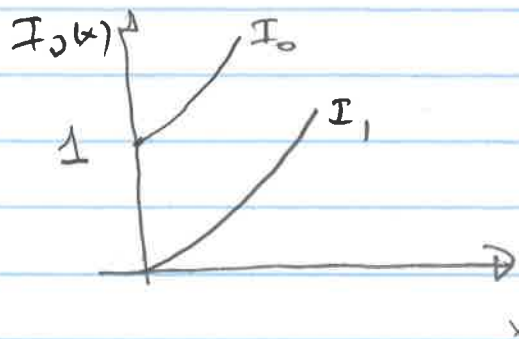
$J_n(x)$ enter
 $\nabla^2 \psi = -k^2 \psi$

$$e^{x/2 (t + 1/t)} = \sum I_n(x) t^n$$

$I_n(x)$ enter
 $\nabla^2 \psi = +k^2 \psi$

$$J_\nu(x) = \sum_s \frac{1}{s!(s+\nu)!} (-1)^s \left(\frac{x}{2}\right)^{2s+\nu}$$

$$I_\nu(x) = \sum_s \frac{1}{s!(s+\nu)!} \left(\frac{x}{2}\right)^{2s+\nu}$$



$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots$$

Analytically $I_\nu(x) = J_\nu(ix)$

$$\cosh x = \cos(ix)$$

Trigonologues

$$e^{ix} = \cos x + i \sin x$$

$$\sinh x = \cosh x$$

expand sines/cosines

$$I_\nu \quad J_\nu$$

SM-2

$$Z = \int_0^{2\pi} d\theta_1 M^N(\theta, \theta_1)$$

??
↑

Trace M^N

Eigenvalues of $M \equiv A$

$$Z = \sum_{\lambda} \lambda^N = \lambda_{\max}^N \sum_{\lambda} \left(\frac{\lambda}{\lambda_{\max}} \right)^N$$

$\rightarrow \lambda_{\max}^N$

Eigenvalues ?

$$\int_0^{2\pi} M(\theta, \theta_1) f(\theta_1) d\theta_1 = \lambda f(\theta)$$

ser
pay
SM-2A!

$$e^{\beta k \cos(\theta - \theta_1)} = \sum_{n=-\infty}^{\infty} I_n(\beta k) e^{in(\theta - \theta_1)}$$

$$\sum_{n=-\infty}^{\infty} I_n(\beta k) \int_0^{2\pi} e^{in(\theta - \theta_1)} f(\theta_1) d\theta_1 = \lambda f(\theta)$$

$$\sum_{n=-\infty}^{\infty} I_n(\beta k) e^{in\theta} \int_0^{2\pi} e^{-in\theta_1} f(\theta_1) d\theta_1 = \lambda f(\theta)$$

??
who

$$f(\theta) = ??$$

$$f(\theta) = e^{im\theta} !!$$

$$\int_0^{2\pi} e^{-in\theta} e^{im\theta} d\theta = 2\pi \delta_{n,m}$$

$$I_n(\beta k) 2\pi e^{im\theta} = \lambda e^{im\theta}$$

$$\lambda = 2\pi I_n(\beta k)$$

$$I_n(x) = \sum \frac{1}{s!(s+n)!} \left(\frac{x}{2}\right)^{2s+n}$$

increasing n decreases $I_n(x)$

$$I_0(x) > I_1(x) > I_2(x) > \dots$$

$$\lambda_{\max} = 2\pi I_0(\beta k)$$

~~$$J_{1/2}(x) = \sum_s \frac{1}{s!(s+\frac{1}{2})!} (-1)^s \left(\frac{x}{2}\right)^{2s+\frac{1}{2}}$$~~