

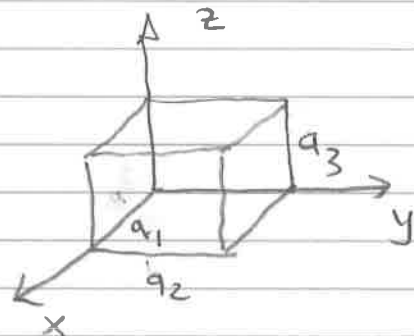
Bessel Function Application to Schroedinger Egn

PROBLEM: Find Energy levels and wave functions of QM particle confined to cylindrical "pillbox".

Lightning review of Rectangular box:

Since $V=0$ inside

$$-\frac{\hbar^2}{2m} \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} \right) \psi = E \psi$$



$$\psi(x, y, z) = A \sin k_1 x \sin k_2 y \sin k_3 z$$

← Automatically ensures $\psi=0$ when $x=0, y=0$, or $z=0$

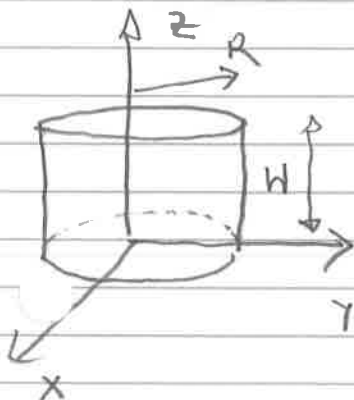
$$k_i a_i = n_i \pi \quad \leftarrow \text{to ensure}$$

$\psi=0$ on $x=a_1$
 $y=a_2$ faces
 $z=a_3$

$$\text{So } E = \frac{\hbar^2}{2m} \pi^2 \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \frac{n_3^2}{a_3^2} \right)$$

n_1, n_2, n_3 integers

For cylindrical geometry:



$$-\frac{\hbar^2}{2m} \left(\frac{d^2}{dp^2} + \frac{1}{p} \frac{d}{dp} + \frac{1}{p^2} \frac{d^2}{d\phi^2} + \frac{d^2}{dz^2} \right) \psi(p, z, \phi) = E \psi$$

$$\psi(p, z, \phi) = A(p) \sin \frac{k_z z}{H} e^{in\phi}$$

as before $k = \frac{k_z}{H}$

BS-2

$$-\frac{\hbar^2}{2m} \left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} n^2 - \frac{l^2 \pi^2}{H^2} \right] A(\rho) e^{in\phi} \sin \frac{l\pi z}{H} = EA(\rho) e^{in\phi} \sin \frac{l\pi z}{H}$$

$$\text{Thus } \left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} n^2 - \frac{l^2 \pi^2}{H^2} + \frac{2mE}{\hbar^2} \right] A(\rho) = 0$$

$$\left[\rho^2 \frac{\partial^2}{\partial \rho^2} + \rho \frac{\partial}{\partial \rho} + (k^2 \rho^2 - n^2) \right] A(\rho) = 0$$

$$k^2 \equiv \frac{2mE}{\hbar^2} - \frac{l^2 \pi^2}{H^2} \quad \therefore A(\rho) = J_n(k\rho)$$

We need $J_n(kR) = 0$

so kR must be a zero of J_n

This quantizes the values of k : $k = k_{nj}$
 \uparrow
 j th zero of J_n

$$E_{njl} = \frac{\hbar^2}{2m} \left[k_{nm}^2 + \frac{l^2 \pi^2}{H^2} \right]$$

$$\psi_{njl}(\rho, \phi, z) = J_n(k_{nj}\rho) \sin \frac{l\pi z}{H} e^{in\phi}$$

BS-3

zeros of Bessel functions

	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$
1	2.4048	3.8317	5.1356	6.3802
2	5.5201	7.0156	8.4172	9.7610
3	8.6537	10.1735	11.6198	13.0152
4	11.7915	13.3237	14.7960	16.2235