DE-4

Do SQUARE $\square$ first

No 11.1.28

Diffusion Eqn - Critical Mass

We wrote down diffusion eqn in absence of any sources

$$D \nabla^2 \psi = \frac{\partial \psi}{\partial t}$$ { Fourier soln

Stuff (ink, temperature, ...) just spreads out.

Like free particle Sch Eqn

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = \frac{\hbar}{i} \frac{\partial \psi}{\partial t}$$ { free propagator, path integral

Except for "i". Now add a source term

$$D \nabla^2 \psi + \lambda \psi = \frac{\partial \psi}{\partial t}$$

If $\psi(r,t) = \psi(t)$ (no spatial dependence)

$$\psi(t) \sim e^{\lambda t}$$ exponential growth/decay

[What about Sch. case: What does λ term correspond to? Does it affect physics?]

HEAT

SEE DE-4

Problem: ~~Melting~~ Bar with internal heat source
have a given amount of material. What cross section most likely to melt $\square$ vs $\bigcirc$?

At end: More dramatic example: neutron diffusion in uranium rod assume $\psi(\vec{r},t) = u(\vec{r})\, f(t)$ ← $e^{-\alpha t}$

$$D \nabla^2 u + (\lambda + \alpha) u = 0$$

<u>Infinite cylinder</u> $u(\vec{r}) = u(\rho, z, \phi) = u(\rho)$

Cylindrical coordinates
$$h_1 = \rho \quad \phi$$
$$h_2 = 1 \quad \rho$$
$$h_3 = 1 \quad z$$

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1} \frac{h_2 h_3}{h_1} \frac{\partial}{\partial q_1} + \cdots\right]$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \phi}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \frac{\partial}{\partial z}\rho\frac{\partial}{\partial z} + \frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}\right]$$

no ϕ, z dependence $\Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}$

$$\Big[D\Big(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\Big) + \underbrace{(\lambda + \alpha)}_{\equiv D k^2}\Big] u = 0$$

$\lambda + \alpha = D k^2$

$\alpha = D k^2 - \lambda$

$$\left[\rho^2 \frac{\partial^2}{\partial \rho^2} + \rho\frac{\partial}{\partial \rho} + k^2\rho^2\right] u = 0$$

$$\left[x^2 \frac{d^2}{dx^2} + x\frac{d}{dx} + x^2 - n^2\right] y = 0 \qquad y = J_n(x)$$

$\therefore \quad u = J_0(k\rho)$

analog of e^{ikx} in $d=1$ or $\cos kx$ cos $k_m y$ on next page

$$u(\rho, \phi, z, t) = \int dk\, a(k)\, J_0(k\rho)\, e^{-(Dk^2 - \lambda)t}$$

Example:

$\left(x\frac{d}{dx} + x\right) y = 0$

~~$\left(\frac{d}{dx} + 1\right) y = 0$~~ skip

~~$dy/dx = -y$~~

$y = e^{-x}$

$\left(x\frac{d}{dx} + kx\right) y = 0$

$y = e^{-kx}$

$dy/dx = -k e^{-kx}$

Yes! $\int_0^\infty J_\nu(\alpha\rho)\, J_\nu(\alpha'\rho) = \frac{1}{\alpha}\delta(\alpha - \alpha')$ for any ν!!

What question naturally arises? Are $J_0(k\rho)$ complete? Is Bessel Hermitian? Involves weight function

DE-3.

temperature

neutron density = 0 at $\rho = a$ — diameter of cylinder

Ka = roots of J_0

not all be allowed! $\left\{ \quad u(\rho,\phi,z,t) = \sum_n a_n J_0(k_n\rho)\, e^{-(Dk_n^2-\lambda)t} \right.$

$\rightarrow$ k_n obeying $J_0(k_n a) = 0$

Exponential growth occurs when

$$\lambda > Dk_n^2$$

$$Dk_n^2 - \lambda < 0$$

again λ large is bad or D small is bad.

$k_n a = 2.405$

$= 5.520$

$= 8.654$

$$k_n^2 < \lambda/D$$

$$(2.405)^2/a^2 < \lambda/D$$

Exponential growth occurs first at smallest root

$$a^2 > (2.405)^2 D/\lambda$$

$$a_{critical} = 2.405\sqrt{\frac{D}{\lambda}}$$

depends on diffusion rate of neutrons (a increases)
and λ rate of production of neutrons (a decreases)

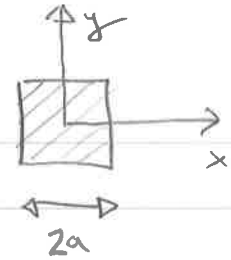
Interesting points ① What happens at $t=\infty$ only $T=0$ or $T=\infty$
no nontrivial steady state?! Answer
Boundary conditions maybe should be $\partial T/\partial x$ not T

② Can A_{00} every be zero?
No, see ~~next~~ rectangle example especially
Initial Temp ≥ 0 at all points so A_{00} cannot vanish

so melting is indep of initial conditions (all IC have $A_{00} \neq 0$)

DE-4

Square cross section



$$D\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\lambda\right)\psi = \frac{\partial \psi}{\partial t}$$

$$\psi = u(r)\, e^{-\alpha t}$$

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{\lambda+\alpha}{D}\right) u(x,y) = 0$$

$k_n a = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$$u(x,y) = \cos k_n x \cos k_m y$$

$\cos(k_n a) = 0$

$k_n = \frac{\pi, 3\pi, 5\pi, \ldots}{2a} = \frac{\pi}{2a}(1, 3, 5, \ldots$

$$-k_n^2 - k_m^2 + \frac{\lambda+\alpha}{D} = 0$$

$$\alpha = -\lambda + D(k_n^2 + k_m^2)$$

$$= -\lambda + D\frac{\pi^2}{4a^2}\left[(2n+1)^2 + (2m+1)^2\right]$$

$$= -\lambda + \frac{D\pi^2}{4a^2}(1+1)$$

$$-\lambda + D\frac{\pi}{4a^2}(1+9)$$

if λ small okay

if "a" small we're okay. If D is large this is large + and $\alpha > 0$: decay

Bad news when $\alpha < 0$. Easiest to occur at $n = m = 0$

$$\lambda = D\frac{\pi^2}{2a^2}$$

$$a^2 = \frac{D\pi^2}{2\lambda} \Rightarrow a = \frac{\pi}{\sqrt{2}}\sqrt{\frac{D}{\lambda}}$$

$$A^{circle}_{crit} = \pi a^2 = (2.405)^2 \pi\, D/\lambda = 18.2\, D/\lambda$$

$$A^{rect}_{crit} = (2a)^2 = \frac{4\pi^2}{2}\frac{D}{\lambda} = 19.7\, D/\lambda$$

radiators

DE-5

$$\int dk \sum_{\ell m} Y_{\ell m}(\phi,\theta)\,\underbrace{\frac{J_{\ell+1/2}(kr)}{\sqrt{r}}}_{r^{\ell},\ r^{-(\ell+1)}}\,A_{\ell m}(k)$$

Sphere of Uranium

$$\left[\nabla^2 + \frac{(\lambda+\alpha)}{D}\right] u(r,\theta,\phi) = 0$$

arbitrary coefficients

$$\psi(r,\theta,\phi,t) = \sum_{\ell, n} A_{\ell n} P_\ell(\cos\theta)\,\frac{1}{\sqrt{r}}\,J_{\ell+\frac{1}{2}}(\underbrace{k_{\ell n}}_{\text{two indices}} r)\,e^{-(Dk_{\ell n}^2-\lambda)t}$$

$k^2 = \dfrac{\lambda+\alpha}{D}$

no ϕ dependence $Y_\ell^m(\theta,\phi) \to P_\ell(\cos\theta)$

Legendre polynomials

$\alpha = Dk^2 - \lambda$

$J_{\ell+\frac{1}{2}}\left[k_{\ell n}\, a\right] = 0$

radial eqn $\sim$ Bessel eqn with $n = \ell + 1/2$

with $1/\sqrt{r}$ in front

$J_{1/2}$

Smallest $k_{\ell n}$ is for ~~$\ell + 1/2$~~ 3.1416 (Abramowitz + Stegun 467)

back of preceding

$$\left(D\frac{\pi^2}{a^2} - \lambda\right) < 0 \qquad \frac{D\pi^2}{a^2} < \lambda \qquad \frac{a^2}{\pi^2 D^2} > \frac{1}{\lambda}$$

$$a > \pi\sqrt{\frac{D}{\lambda}}$$

$$V = \frac{4}{3}\pi a^3 = \frac{4}{3}\pi\ \pi^3\left(\frac{D}{\lambda}\right)^{3/2} = \frac{4\pi^4}{3}\left(\frac{D}{\lambda}\right)^{3/2}$$

Hemisphere $\ell + \frac{1}{2} = 3/2$ 4.493 only odd ℓ allowed

$P_\ell(\cos\pi/2) = 0$

$P_\ell(0) = 0$

$$\left.\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \tfrac{3}{2}x^2 - \tfrac{1}{2} \end{aligned}\right\}$$

$a > 4.493\sqrt{\dfrac{D}{\lambda}}$ bigger as expected!

$$\left(\frac{4.493}{3.142}\right)^3 = 2.925$$