PHYSICS 200B, WINTER 2017
ELECTRICITY AND MAGNETISM

Assignment Five Due Monday, March 13, 5:00 pm.

[1.] Verify that $(\mu_0\,/\,4\pi)\,\mathbf{m}\times\mathbf{r}/r^3$ is a suitable vector potential for a point magnetic dipole, i.e. that its curl gives a dipole magnetic field. Assume $r \neq 0$.

[2.] Find the vector potential and magnetic field of an infinite straight wire of cross section with radius a, carrying a current I uniformly distributed across its area. (A good starting point is the thin wire $a \to 0$ done in class.)

[3.] When a type-II superconductor is placed in a magnetic field, the field becomes non-uniform and passes through the superconductor in thin cylindrical regions called *flux tubes* or *vortices*. (See figure at left below.) Here we study a toy model of a flux tube. Take $\mathbf{B}$ parallel to $\hat{\mathbf{z}}$ everywhere, the z axis along the axis of the tube, and $r_\perp = (x^2+y^2)^{1/2}$. Assume the magnetic flux through a circle of radius $r_\perp$ centered on the axis of the tube is

$$\Phi(r_\perp) = 2\Phi_0 \frac{r_\perp^2}{r_\perp^2 + a^2} \qquad\qquad (\Phi_0, a \text{ are constants}).$$

Find $\mathbf{A}(\mathbf{r})$, $\mathbf{B}(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$ accompanying the tube. Make informatively marked and labelled sketches showing the spatial variation of $\mathbf{B}, \mathbf{j}$ and Φ.

[4.] A circular wire loop of radius R_1 carrying current I_1 is placed in the xy plane with its center at the origin. A second circular wire loop of radius R_2, carrying current I_2 lies parallel to the first one, with its center at $(0, 0, h)$. Assume the first loop is very small so that $R_1 \ll h$ and $R_1 \ll R_2$. See figure at right below.

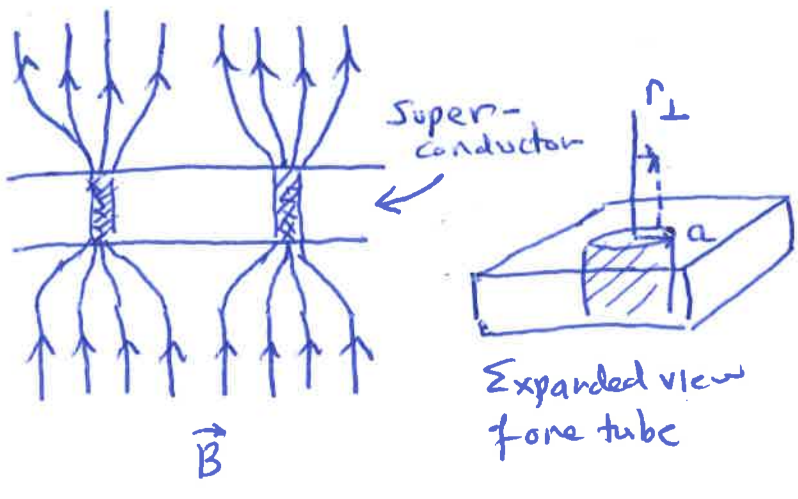
Show that the field due to the upper loop is given by

$$B_z = \frac{\mu_0}{4\pi}\frac{2\pi R_2^2 I_2}{R_2^2 + (z-h)^2}$$

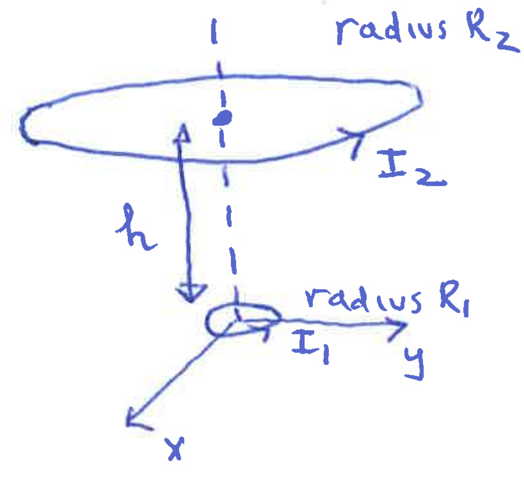
at points along the z axis. Using the fact that R_1 is very small, compute the magnetic flux Φ_{12} through the small loop due to the field created by the big loop. Find the mutual inductance $L_{12} = \Phi_{12}/I_2$.

Using a physically equivalent set of approximations, find the magnetic flux Φ_{21} through the big loop due to the field created by the small loop. Compute L_{21} and show $L_{21} = L_{12}$. (*Do not use this theorem to do the problem!*)

PROBLEM 3:



PROBLEM 4:



$\boxed{5-1}$

1. Magnetic Dipole vector potential:

Recall that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

We want $\vec{B} = \vec{\nabla} \times \left(\vec{m} \times \frac{\vec{r}}{r^3}\right)$, so $\vec{b}$ is a constant vector, and

$$\vec{B} = \vec{m}\underbrace{\left(\vec{\nabla}\cdot\frac{\vec{r}}{r^3}\right)}_{\text{E-field of pt. charge}} - (\vec{m}\cdot\vec{\nabla})\frac{\vec{r}}{r^3}$$

Ignoring δ-fn at origin, we know $\vec{\nabla}\cdot\vec{E} = 0$,

and $(\vec{m}\cdot\vec{\nabla})\vec{r} = \vec{m}$

$$(\vec{m}\cdot\vec{\nabla})\frac{1}{r^3} = \vec{m}\cdot\left(-\frac{3\vec{r}}{r^5}\right)$$

$$\therefore \quad \vec{B} = \frac{3(\vec{m}\cdot\vec{r})\vec{r} - \vec{m}r^2}{r^5}, \quad W^5$$

2. $$\vec{\nabla} \times \underbrace{B_0 x \hat{y}}_{\vec{A}} = \hat{z}\,\frac{\partial}{\partial x}A_y = B_0\hat{z}$$

$$\vec{\nabla} \times (-B_0 y\hat{x}) = \hat{z}\left(-\frac{\partial}{\partial y}A_x\right) = B_0\hat{z}$$

Add & divide by 2

$$\vec{\nabla} \times \frac{1}{2}B_0(x\hat{y} - y\hat{x}) = B_0\hat{z}.$$

$$-B_0 y\hat{x} = B_0 x\hat{y} - \vec{\nabla}(B_0 x y)$$

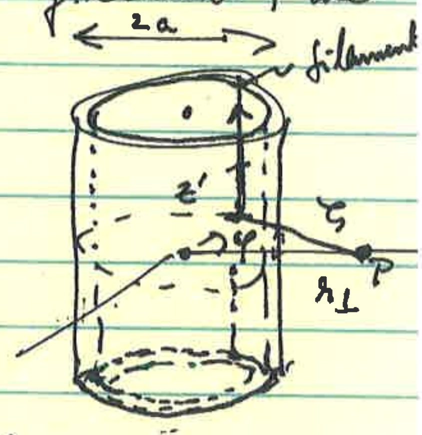
$$= \frac{1}{2}B_0(x\hat{y} - y\hat{x}) - \vec{\nabla}\left(\frac{B_0 xy}{2}\right).$$

QED

<u>Vector potential for a thick current carrying wire</u>

For a thin/mathematical wire or filament, we found

$$A_z = -\frac{2I}{c} \ln \frac{r_\perp}{2L}, \quad (1)$$



where L was the regulatory cutoff,

and $A_x = A_y = 0$.

Hollow pipe

From the filaments, let us synthesize a conductor in the form of a hollow pipe of radius a. If the total current is I, the current through a segment of angular width $d\varphi$ is $(I\, d\varphi / 2\pi)$. So in Eq. (1),

$$\left. \begin{aligned} I &\to I\, d\varphi / 2\pi \\ r_\perp &\to \xi(r_\perp, a, \varphi) = \left(a^2 + r_\perp^2 - 2 a r_\perp \cos\varphi\right)^{1/2}, \end{aligned} \right\} ($$

and we must integrate over φ. Thus, for the pipe,

$$A_z = -\frac{2I}{2\pi c} \int_0^{2\pi} \underbrace{\ln \frac{\xi}{2L}}_{} \, d\varphi \quad (3).$$

$$= \frac{1}{2} \ln \frac{\xi^2}{a^2} + \ln \frac{a}{2L} \quad (4)$$

$$= -\frac{I}{\pi c} \left[2\pi \ln \frac{a}{2L} + \int_0^{\pi} \ln \frac{\xi^2}{a^2} \, d\varphi \right] \quad (5)$$

Note

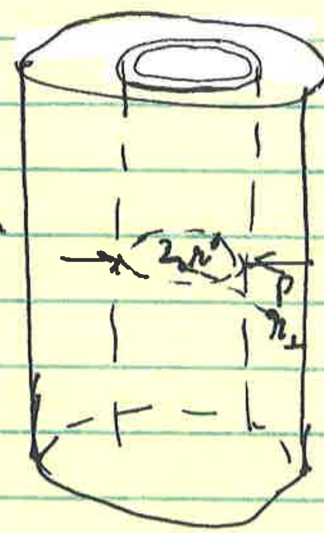
The integral is precisely the one we saw while finding the electrostatic potential of a charged hollow pipe (Notes of 2/17/14; Eq. (10).) Reusing the result, we get

$$A_z = -\frac{2I}{c}\left[\ln\frac{a}{2L} + \begin{cases} 0 \,, & r_\perp < a \\ \ln\frac{r_\perp}{a} \,, & r_\perp > a\,. \end{cases}\right]$$

$$= -\frac{2I}{c} \times \begin{cases} \ln(a/2L) \,, & r_\perp < a \\ \ln(r_\perp/2L) \,, & r_\perp > a \end{cases} \,. \quad (6)$$

Note how the form for $r_\perp > a$ is identical to (1).

Thick wire of radius a.



Build wire out of pipes. If total current is I, current through pipe of radius r' to $r' + dr'$ is

$$(I/\pi a^2)(2\pi r'\,dr')\,. \quad (7)$$

So in Eq. (6),

$$\left.\begin{aligned} I &\to \frac{2I}{a^2}\, r'\,dr' \\ a &\to r' \end{aligned}\right\} (8)\,,$$

and we must integrate over r'.

If $r_\perp > a$,

$$A_z = -\frac{4I}{a^2 c}\int_0^a \left(\ln\frac{r_\perp}{2L}\right) r'\,dr' = -\frac{2I}{c}\ln\frac{r_\perp}{2L}\,; \quad (9$$

If $r_\perp < a$

$$A_z = -\frac{4I}{a^2 c}\left[\int_0^{r_\perp} \ln\frac{r_\perp}{2L}\, r'\,dr' + \int_{r_\perp}^{a} \ln\frac{r'}{2L}\, r'\,dr'\right] \quad (10$$

Do the second integral by parts:

$$\int_{r_\perp}^{a} (\ln r')\, r'\, dr' = (\ln r') \frac{r'^2}{2}\Bigg|_{r_\perp}^{a} - \int_{r_\perp}^{a} \frac{r'^2}{2} \cdot \frac{1}{r'}\, dr'$$

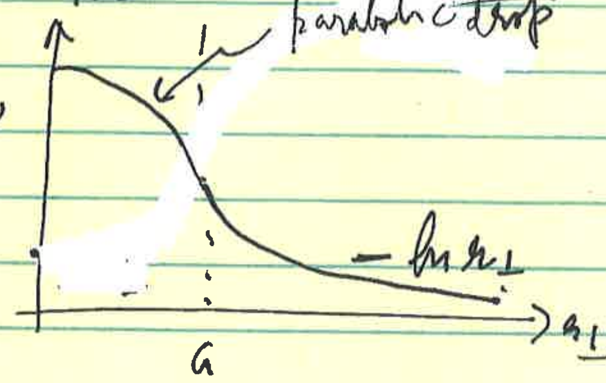
$$= \frac{1}{2}\left(a^2 \ln a - r_\perp^2 \ln r_\perp\right) - \frac{1}{4}\left(a^2 - r_\perp^2\right). \qquad (11)$$

Adding in the first term in (10), we get

$$A_z = -\frac{4I}{a^2 c}\left[\frac{1}{2} a^2 \ln \frac{a}{2L} - \frac{1}{4}\left(a^2 - r_\perp^2\right)\right]$$

$$= -\frac{2I}{c} \ln \frac{a}{2L} + \frac{I}{c}\left(1 - \frac{r_\perp^2}{a^2}\right). \quad (r_\perp < a) \quad (12$$

Note how (12) matches smoothly onto (9), and how (9) has exactly the same form as the thin wire and hollow pipe outside the pipe - A_z



To find the magnetic field, we use $B_\varphi = -\partial A_z / \partial r_\perp$, obtaining

$$\vec{B} = \frac{2I}{c} \times \begin{cases} \dfrac{r_\perp}{a^2}\, \hat{e}_\varphi, & r_\perp < a \\ \dfrac{1}{r_\perp}\, \hat{e}_\varphi, & r_\perp > a. \end{cases} \qquad (13)$$

QED

Note similarity with $\vec{E}$-field of uniformly charged, insulating thick rod.

$\boxed{5-3}$

(a) $\displaystyle\oint_{C=\partial S} \vec{A}\cdot d\vec{l} \underset{\uparrow}{=} \int_S (\nabla\times\vec{A})\cdot d\vec{s} = \int_S \vec{B}\cdot d\vec{s} = \Phi$ (flux through C).

by Stokes theorem

The flux is physically measurable and gauge invariant.

(b) integrate $\vec{A}$ around a circle of radius $r_\perp$ assuming $\vec{A}$ is cylindrically symmetric.

Then $\displaystyle\oint_C \vec{A}\cdot d\vec{l} = A_\varphi(r_\perp)(2\pi r_\perp)$. (1)

Equating this to $\Phi(r_\perp)$ yields $\displaystyle A_\varphi(r_\perp) = \frac{\Phi_0}{\pi}\frac{r_\perp}{r_\perp^2 + a^2}$, (2)

or $\displaystyle\vec{A}(\vec{x}) = \frac{\Phi_0}{\pi}\frac{(\hat{z}\times\vec{r})}{r_\perp^2+a^2} = \frac{\Phi_0}{\pi}\frac{\hat{z}\times\vec{r}}{\ldots}$ (3)

$$\nabla \times \mathbf{v} = \hat{\mathbf{e}}_\perp \left(\frac{1}{r_\perp} \frac{\partial v_z}{\partial \varphi} - \frac{\partial v_\varphi}{\partial z} \right) + \hat{\mathbf{e}}_\varphi \left(\frac{\partial v_\perp}{\partial z} - \frac{\partial v_z}{\partial r_\perp} \right) + \hat{\mathbf{e}}_z \frac{1}{r_\perp} \left(\frac{\partial}{\partial r_\perp}(r_\perp v_\varphi) - \frac{\partial v_\perp}{\partial \varphi} \right).$$

$$\vec{A} = \frac{\Phi_0}{\pi} \frac{(x\hat{y} - y\hat{x})}{(x^2+y^2+a^2)} \qquad (4)$$

$\vec{B} = \vec{\nabla} \times \vec{A}$. In this case $\vec{B} \parallel \hat{z}$ and depends only on $r_\perp$, so can find it by direct evaluation of the curl, or by noting that

$$\Phi(r_\perp) = \int_0^{r_\perp} B_z(r'_\perp)\, 2\pi r'_\perp\, dr'_\perp . \qquad (5)$$

Differentiate: $\quad d\Phi(r_\perp)/dr_\perp = 2\pi r_\perp B_z(r_\perp); \qquad (6)$

$$\boxed{B_z(r_\perp)} = \frac{1}{2\pi r_\perp} \frac{d}{dr_\perp} \Phi(r_\perp) = \frac{\cancel{2}\Phi_0}{\cancel{2}\pi r_\perp} \left(\frac{2r_\perp}{r_\perp^2 + a^2} - \frac{2r_\perp^3}{(r_\perp^2+a^2)^2} \right)$$

$$\boxed{= \frac{2\Phi_0}{\pi} \frac{a^2}{(r_\perp^2+a^2)^2}} . \qquad (7)$$

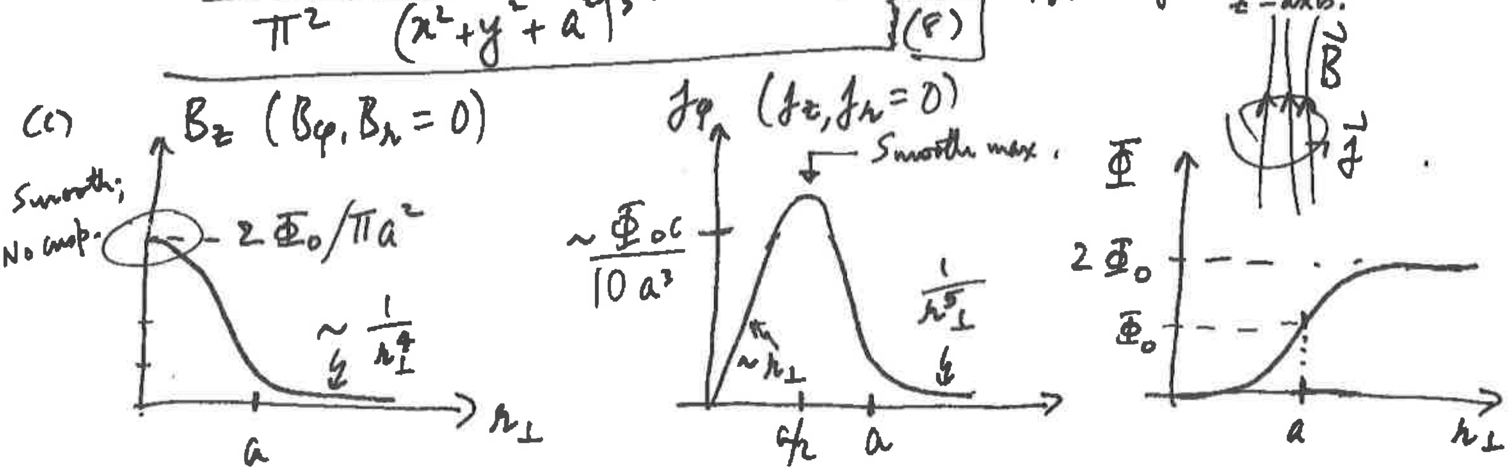
Finally, $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$, so

$$\vec{j} = \frac{c}{4\pi} \vec{\nabla} \times \frac{2\Phi_0}{\pi} \frac{a^2}{(x^2+y^2+a^2)^2} \hat{z} = -\frac{\Phi_0 c}{2\pi^2} \hat{z} \times \vec{\nabla} \frac{a^2}{(x^2+y^2+a^2)^2} \quad \text{(in Gaussian)}$$

$$\boxed{= \frac{2\Phi_0 c}{\pi^2} \frac{a^2}{(x^2+y^2+a^2)^3} (\hat{z} \times \vec{r})} . \qquad \boxed{(8)}$$

so $\vec{j}$ circles around z-axis.

(c)



$\boxed{5-4}$

(a) Since $R_1 \ll R_2, h$, assume field of loop 2 is uniform over the area of the small loop, i.e., that its variation can be neglected. Then

$$\Phi_{1,2} \approx \pi R_1^2 \cdot B_z(0) = \pi R_1^2 \cdot \frac{2\pi R_2^2}{c\,(R_2^2+h^2)^{3/2}} I_2 = \frac{2\pi R_1^2 R_2^2}{c\,(R_2^2+h^2)^{3/2}} I_2 \quad (1)$$

$$L_{12} = \Phi_{1,2}/cI_2 = \frac{2\pi R_1^2 R_2^2}{c^2(R_2^2+h^2)^{3/2}} \quad (2)$$

(b) Since $R_1 \ll R_2$, we can treat loop 1 as a point dipole to find its field.

$$\text{dipole moment} \equiv \vec{m} = (\pi R_1^2 I_1/c)\,\hat{z} \quad (3)$$

$$\vec{B}(\vec{r}) = \frac{1}{r^5}\left(3(\vec{m}\cdot\vec{r})\vec{r} - \vec{m}r^2\right). \quad (4)$$

$$\Phi_{1,2} = \int_{S_2} \vec{B}(\vec{r})\cdot d\vec{s} = \iint_{r_\perp < R_2} B_z(r_\perp, z=h)\, d^2s \quad \left(\begin{array}{c}\text{Take } S_2 \text{ as a flat}\\ \text{disk of radius } R_2.\end{array}\right) \quad (5)$$

$$\left(\text{Note: } r_\perp = \sqrt{x^2+y^2} \neq r\right)$$

$$= \int_0^{R_2} \frac{3mh^2 - m(r_\perp^2+h^2)}{(r_\perp^2+h^2)^{5/2}}\, 2\pi r_\perp\, dr \quad (6)$$

$$= \pi m \left[\frac{2}{3}\times\frac{3h^2}{(r_\perp^2+h^2)^{3/2}} - \frac{2}{1}\times\frac{1}{(r_\perp^2+h^2)^{1/2}}\right]_{r_\perp=R_2}^{r_\perp=0} = \frac{2\pi m R_2^2}{(R_2^2+h^2)^{3/2}} \quad (7)$$

Feed in (3) for m; divide by $I_1 c$:

$$L_{21} = \frac{1}{c\cancel{I_1}} \cdot \frac{2\pi^2 R_1^2 R_2^2}{(R_2^2+h^2)^{3/2}}$$