PHYSICS 200B, WINTER 2017
ELECTRICITY AND MAGNETISM

Assignment Four, Due Monday, February 27, 5:00 pm.

[1.] Solve for the potential inside a cylindrical box $0 < z < L$ and $0 < \rho < b$ given that the potential vanishes at the top and bottom of the box $V(\rho, \phi, z = 0) = V(\rho, \phi, z = L) = 0$ and takes an arbitrary functional form $V(\rho = b, z, \phi)$ along the box sides. You will, of course, have to leave your answer in the form of integrals over the unspecified function $V(\rho = b, z, \phi)$.

[2.] Apply your solution to problem [1.] to the particular case $V(\rho = b, z, \phi) = +V_0$ for $-\pi/2 < \phi < +\pi/2$ and $V(\rho = b, z, \phi) = -V_0$ for $\pi/2 < \phi < +3\pi/2$. That is, evaluate the coefficients, now that you know the specific potential along the side walls.

[3.] Compute the partition function of the 1D XY model. (We will do the Ising model, and start this XY problem out in class Wednesday, so you can see what is involved here.)

Physics 200B Homework 4 Solns
Winter 2017

In cylindrical coordinates we write

$$V(\rho, z, \phi) = R(\rho)\, Q(\phi)\, T(z)$$

and insert in the Laplace Eqn

primes mean derivatives with respect to appropriate variable

$$0 = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= R'' Q T + {}^1\!/_\rho\, R' Q T + {}^1\!/_{\rho^2}\, R Q'' T + R Q T''$$

Letting $T'' = -k^2 T$ and $Q'' = -\nu^2 Q$ yields

or $\sin kz$, $\cos kz$, ... $\rightarrow$ $T(z) = e^{\pm ikz}$ $\quad Q(\phi) = e^{\pm i\nu\phi}$ $\leftarrow$ Actually $\nu = m$ an integer because $Q(\phi + 2\pi) = Q(\phi)$

and $\quad 0 = R'' + {}^1\!/_\rho\, R' - {}^1\!/_{\rho^2}\, \nu^2 R - k^2 R$

Defining $x = k\rho$ and also R' now meaning d/dx

$$R'' + \frac{1}{x} R' + \left(-1 - \frac{\nu^2}{x^2}\right) R = 0 \qquad \nu = m \text{ (integer)}$$

The solns to this eqn that are well behaved at the origin $x \to 0$ ($\rho \to 0$) are $I_m(x)$

The conditions that $V=0$ at $z=0$ and $z=0$ restrict

$$e^{ikz} \rightarrow \sin n\pi z/L$$

writing $e^{im\phi}$ as sines and cosines and assembling everything

$$V(\rho,z,\phi) = \sum_m \sum_n \left(A_{mn} \sin m\phi + B_{mn} \cos m\phi\right) \sin \frac{n\pi z}{L} \, I_m\left(\frac{n\pi \rho}{L}\right)$$

If we are <u>given</u> $V(\rho=b,z,\phi)$ along the cylinder sides

we can determine the coefficients A_m and B_m

$$V(b,z,\phi) = \sum_{mn} \left(A_{mn} \sin m\phi + B_{mn} \cos m\phi\right) \sin \frac{n\pi z}{L} \, I_m\left(\frac{n\pi b}{L}\right)$$

by the usual Fourier inversion (aka orthogonality of Fourier series)

$$I_{m'}\left(\frac{n'\pi b}{L}\right) A_{m'n'} = \int_0^{2\pi} \frac{d\phi}{\pi} \sin m'\phi \int_0^L \frac{dz}{L/2} \sin \frac{n'\pi z}{L} \, V(b,z,\phi)$$

or, redefining $m' \rightarrow m$ $n' \rightarrow n$

$$A_{mn} = 1/I_m\left(\frac{n\pi b}{L}\right) \; \frac{2}{L\pi} \int_0^{2\pi} d\phi \int_0^L dz \; \sin m\phi \sin \frac{n\pi z}{L} \, V(b,z,\phi)$$

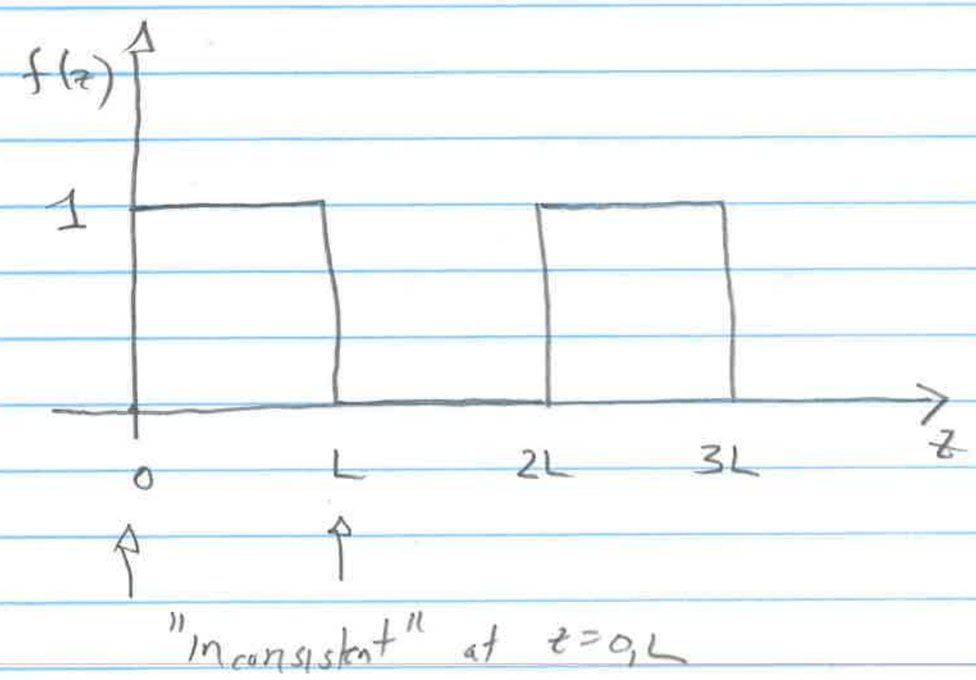
with a similar expression for B_{mn} replacing $\sin m\phi$ by $\cos m\phi$

2-1

We are asked to apply problem #1 to

$$V(b,z,\phi) = \begin{cases} V_0 & -\pi/2 < \phi < \pi/2 \\ -V_0 & \pi/2 < \phi < 3\pi/2 \end{cases}$$

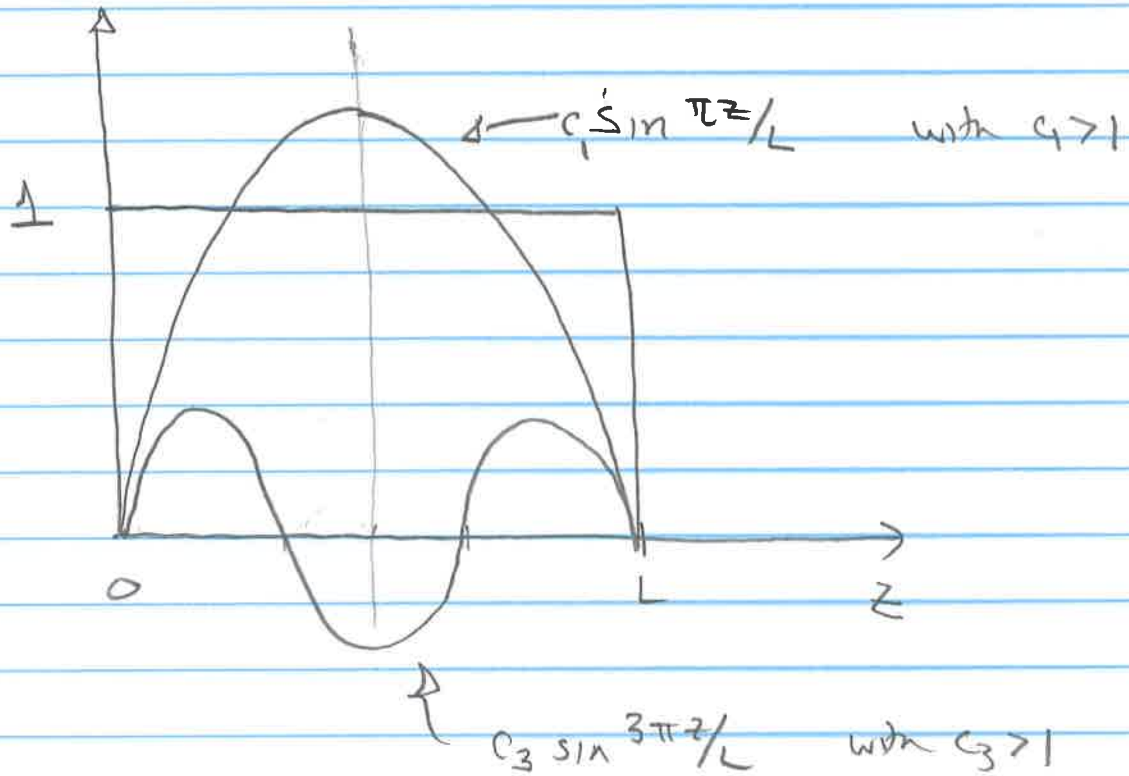
Let's first deal with the z dependence. The problem is a bit poorly stated because the $V(b,z,\phi)$ side boundary conditions are inconsistent with the vanishing of V at the bottom and top of cylinder. But this is a common "inconsistency" which occurs in the Fourier decomposition of a square pulse which also has a discontinuity



2-2

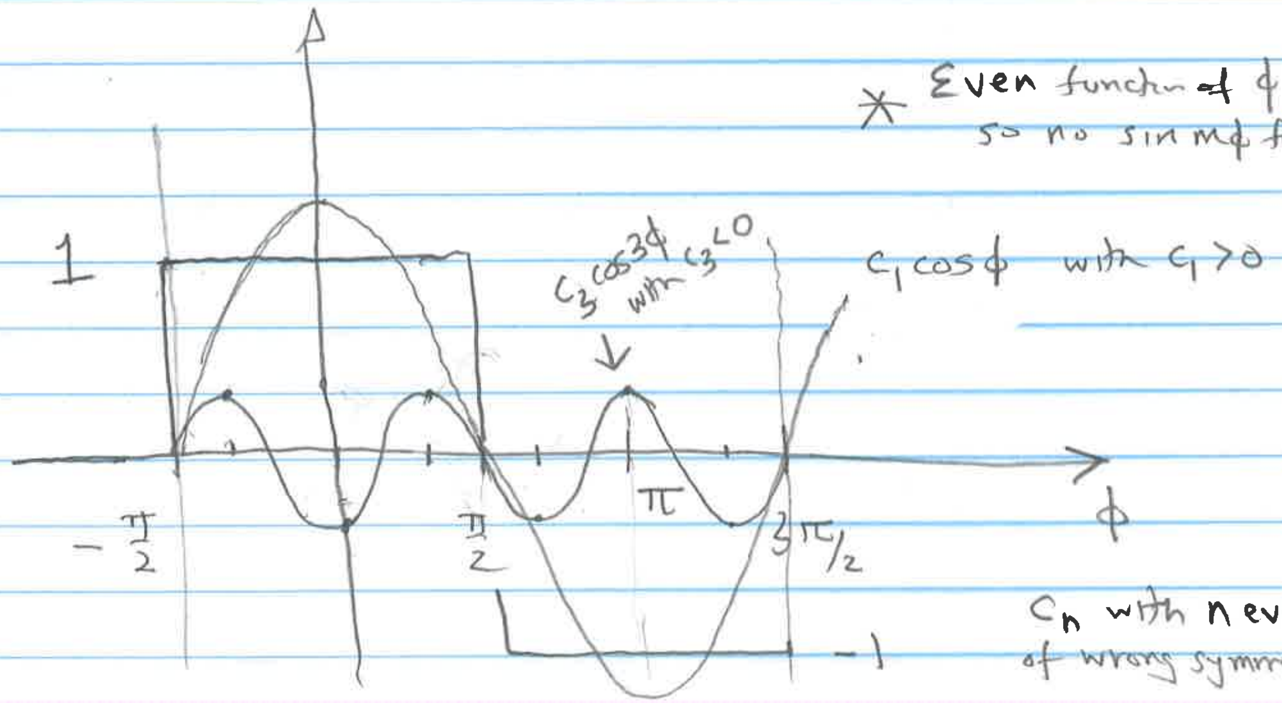
We know how to build up such a square pulse

pictorially:



no c_2 term (wrong symmetry)

Likewise, in the ϕ direction we want



* Even function of ϕ so no $\sin m\phi$ terms

c_n with n even of wrong symmetry

These pictures suggest that in the general sol'n of page 1-2

i) $A_{mn} = 0$ ← $\sin m\phi$ has wrong symmetry

ii) $B_{mn} = 0$ for n even ← $\sin \frac{n\pi z}{L}$ wrong symmetry if n even

iii) $B_{mn} = 0$ for m even ← $\cos m\phi$ wrong symmetry for m even

$$V(\rho, z, \phi) = \sum_{m=1,3,5,..} \sum_{n=1,3,5} B_{mn} \cos m\phi \sin \frac{n\pi z}{L} I_m\left(\frac{n\pi \rho}{L}\right)$$

Suppose we wanted B_{11}, the largest term. We would write

$$I_1\left(\frac{\pi b}{L}\right) B_{11} = \frac{2V_0}{L\pi}\left\{\int_{-\pi/2}^{\pi/2} d\phi \cos\phi - \int_{\pi/2}^{3\pi/2} d\phi \cos\phi\right\} \int_0^L dz \sin\frac{\pi z}{L}$$

$$\downarrow \qquad\qquad \downarrow$$

$$+\sin\phi\Big|_{-\pi/2}^{\pi/2} - \sin\phi\Big|_{\pi/2}^{3\pi/2} \qquad -\frac{L}{\pi}\cos\frac{\pi z}{L}\Big|_0^L$$

$$\downarrow$$

$$+\frac{2L}{\pi}$$

$$+(1-(-1)) - (-1-1)$$

$$\downarrow$$

$$+4$$

coefficient >1 as suggested from pictures!

$$B_{11} = \frac{16V_0}{\pi^2} \, 1/I_1(\pi b/L)$$

$B_{11} > 0$ as expected!

The general expressions are obtained from

$$\int_0^L dz \sin\frac{n\pi z}{L} = -\frac{L}{n\pi} \cos\frac{n\pi z}{L}\Big|_0^L$$

$$= -\frac{L}{n\pi}\left(\cos n\pi - 1\right)$$

$$= \begin{cases} 0 & n \text{ even (as we expected)} \\ +\frac{2L}{n\pi} & n \text{ odd} \end{cases}$$

$$\int_{-\pi/2}^{\pi/2} d\phi \cos m\phi - \int_{\pi/2}^{3\pi/2} d\phi \cos m\phi$$

& notice all n have + contributions see p 2-2 top picture

$$= \frac{1}{m}\left\{\sin m\phi \Big|_{-\pi/2}^{\pi/2} - \sin m\phi \Big|_{\pi/2}^{3\pi/2}\right\}$$

$$= \frac{1}{m}\left\{(-1)^m\right\} 4 \qquad m \text{ odd}$$

m terms oscillate in sign as expected see p 2-2 bottom picture

$$0 \qquad m \text{ even}$$

← as expected

2-5

Thus the general expression

$$B_{mn} = \frac{16 V_0}{\pi^2} \frac{1}{nm} (-1)^m \, 1/I_m\left(\frac{n\pi b}{L}\right)$$

$$Z = \int d\theta_1 d\theta_2 \ldots d\theta_N \; e^{-\beta E}$$

$$E = -J \sum_i \cos(\theta_i - \theta_{i+1})$$

$$Z = \int d\theta_1 d\theta_2 \ldots d\theta_N \; e^{\beta J \sum_i \cos(\theta_i - \theta_j)}$$

$$e^{\beta J \cos(\theta_1 - \theta_2)} \, e^{\beta J \cos(\theta_2 - \theta_3)} \ldots e^{\beta J \cos(\theta_N - \theta_1)}$$

define — assumes pbc

"transfer matrix" $\quad M(\theta_\ell \, \theta_{\ell+1}) \equiv e^{\beta J \cos(\theta_\ell - \theta_{\ell+1})}$

$$\int d\theta_2 \; e^{\beta J \cos(\theta_1 - \theta_2)} \, e^{\beta J \cos(\theta_2 - \theta_3)}$$

$$= \int d\theta_2 \; M(\theta_1 \theta_2) M(\theta_2 \theta_3)$$

$$= M^2(\theta_1 \theta_3)$$ by usual rules of matrix multiplication

Continuing this process

$$\int d\theta_3 \; M^2(\theta_1 \theta_3) M(\theta_3 \theta_4) = M^3(\theta_1 \theta_4)$$

Until, ultimately

$$Z = \int d\theta_1 \; M^N(\theta_1 \theta_1) \Leftarrow \mathrm{tr}\, M^N$$ by usual definition of trace

3-2

For any matrix M^N, $\mathrm{tr}\, M^N$ is most easily computed in basis where M is diagonal. Then

$$\mathrm{tr}\, M^N = \sum_\alpha \lambda_\alpha^N$$

λ_α ← eigenvalues of M.

We seek the eigenvectors (eigenfunctions)

$$\int d\theta_2 \; M(\theta_1, \theta_2) f(\theta_2) = \lambda f(\theta_1)$$

to get $f(\theta)$ we notice

$$M(\theta_1, \theta_2) = e^{\beta J \cos(\theta_1 - \theta_2)} = \sum_n \underbrace{I_n(\beta J)}_{\text{Bessel functions}} e^{in(\theta_1 - \theta_2)}$$

Amazingly, this immediately tells us $f(\theta) = e^{im\theta}$ because

$$\int d\theta_2 \sum_n I_n(\beta J) e^{in(\theta_1 - \theta_2)} e^{im\theta_2}$$

$$= \sum_n I_n(\beta J) e^{in\theta_1} 2\pi \delta_{n,m} = \underbrace{2\pi I_m(\beta J)}_{\lambda} \underbrace{e^{im\theta_1}}_{f(\theta_1)}$$

thus $$Z = \sum_m \left(2\pi I_m(\beta J)\right)^N$$

3-3

It turns out that $I_0 > I_1 > I_2 > \ldots$

so in the limit $N \to \infty$

$$Z = \left[2\pi I_0 / (\beta J)\right]^N$$