PHYSICS 200B, WINTER 2017
ELECTRICITY AND MAGNETISM

Assignment Three, Due Friday, February 17, 5:00 pm.

[1.] A sphere of radius R has a potential $R(r,\theta) = V_0 \cos^2\theta$ on its surface. Determine the potential outside the sphere.

[2.] A sphere of radius R has surface charge density given by $\sigma = \sigma_0 \sin 2\theta \sin\phi$. Determine the potential both inside and outside the sphere. Note: we have not discussed this in class, but it is an easy application of Gauss' Law to relate the discontinuity in the radial derivative of the potential, that is, the radial component of the electric field, to the surface charge density. You will need this.

[3.] Solve for the potential in the region between two concentric spherical shells of radii a and b, given the potentials $V_a(\theta)$ and $V_b(\theta)$. Your objective should be to write the coefficients of an expansion of the potential as integrals involving the (unknown) functions V_a and V_b. Choose some specific form of the functions that you find particularly amusing (and not too hard!) and do the integrals.

1-1

Physics 200B W2017

HW 3 Sol'ns

We know the sol'n of Laplace's eqn with azimuthal symmetry is

$$(*) \quad V(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-(\ell+1)} \right] P_\ell(\cos\theta)$$

Solving outside the sphere allows $r \to \infty$ so we need $A_\ell = 0 \ \forall \ell$.

The B_ℓ are determined by the boundary condition on the sphere's surface. Since

$$P_0(x) = 1$$
$$P_1(x) = x \qquad \Longrightarrow \qquad x^2 = \frac{2}{3} P_2(x) + P_1(x)$$
$$P_2(x) = \tfrac{3}{2}(x^2 - 1)$$

$$V(R,\theta) = V_0 \cos^2\theta = V_0 \left[\frac{2}{3} P_2(\cos\theta) + P_1(\cos\theta) \right]$$

Clearly then

$$V(r,\theta) = V_0 \left(\frac{R}{r}\right)^2 P_1(\cos\theta) + V_0 \frac{2}{3} \left(\frac{R}{r}\right)^3 P_2(\cos\theta)$$

satisfies bdy conditions and is also of the general form $(*)$. Obviously we can easily do any sort of polynomial $V(R,\theta)$ by just decomposing it in $P_\ell(\cos\theta)$.

Lets do the simplest problem of this sort, a shell of constant σ. We know the answer. Outside the shell it behaves like a point charge so

Outside $$V(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^2\sigma}{r} = \frac{1}{\epsilon_0}\frac{R^2\sigma}{r}$$

Inside V = constant because $\vec{E}$ vanishes. to match the formula for $r>R$ we have

Inside $$V(r,\theta,\phi) = \frac{1}{\epsilon_0} R\sigma$$

But let's not use this knowledge and instead proceed from the general forms

$$r>R \qquad V(r,\theta,\phi) = \sum_{\ell m} B_{\ell m} r^{-(\ell+1)} Y_{\ell m}(\theta,\phi)$$

$$r<R \qquad V(r,\theta,\phi) = \sum_{\ell m} A_{\ell m} r^{\ell} Y_{\ell m}(\theta,\phi)$$

At the surface

Gauss' Law:

$r>R$ $\uparrow E_r^+ = -\frac{\partial V}{\partial r}$

$r<R$ $\uparrow E_r^- = -\frac{\partial V}{\partial r}$

$$\phi_E = (E_r^+ - E_r^-)A = \sigma A/\epsilon_0$$

This is a constant and, if decomposed into the independent spherical harmonics, $\therefore$ contains only Y_{00}

$$r>R \qquad E_r^+ = \sum_{\ell m} (\ell+1) B_{\ell m} r^{-(\ell+2)} Y_{\ell m}(\theta,\phi)$$

$$r<R \qquad E_r^- = -\sum_{\ell m} \ell A_{\ell m} r^{\ell-1} Y_{\ell m}(\theta,\phi)$$

We conclude, using the orthogonality of the $Y_{\ell m}$

$$\ell \neq 0 \qquad (\ell+1) B_{\ell m} R^{-(\ell+2)} + \ell A_{\ell m} R^{\ell-1} = 0$$

$$\ell = m = 0 \qquad B_{00} R^{-2} = \sigma/\epsilon_0$$

If we also note V must be continuous at $r=R$

$$\sum_{\ell m} A_{\ell m} R^{\ell} Y_{\ell m}(\theta,\phi) = \sum_{\ell m} B_{\ell m} R^{-(\ell+1)} Y_{\ell m}(\theta,\phi)$$

Using the orthogonality of the $Y_{\ell m}$

$$A_{\ell m} R^{\ell} = B_{\ell m} R^{-(\ell+1)}$$

Combining this with the $\ell \neq 0$ eqn the only possible soln is

$A_{\ell m} = B_{\ell m} = 0$.

So, we are left with

$$r > R \qquad V(r,\theta,\phi) = \frac{B_{00}}{r} = \sigma R^2/\epsilon_0 r$$

recovering the "obvious" soln.

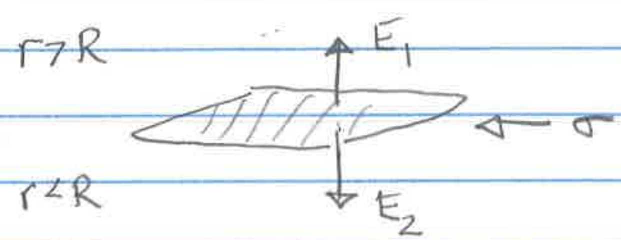
This problem is harder than 1 or 3. The first complication arises from the lack of azimuthal symmetry so that we are forced to write

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^{l} + B_{lm} r^{-(l+1)} \right] Y_{lm}(\theta,\phi)$$

as the solution to Laplace's Eqn

Inside the sphere we have $B_{lm} = 0$ $\forall\, lm$ to avoid a diverging potential at $r=0$. Outside the sphere $A_{lm} = 0$ $\forall\, lm$ to be well defined at $r=\infty$.

The next complication is to understand how σ relates the behavior of V for $r<R$ to that at $r>R$. We use Gauss' law



$$\Phi_E = (E_1 A + E_2 A) = \frac{\sigma A}{\epsilon_0}$$

$$E_1 + E_2 = \sigma/\epsilon_0$$

By E_1 and E_2 I really mean the "radial" part of E, ie the part of $\vec{E}$ which contributes to the flux Φ_E.

$$r<R \qquad V(r,\theta,\phi) = \sum_{\ell m} A_{\ell m} r^{\ell} Y_{\ell m}(\theta,\phi)$$

$$r>R \qquad V(r,\theta,\phi) = \sum_{\ell m} B_{\ell m} r^{-(\ell+1)} Y_{\ell m}(\theta,\phi)$$

The simplest $A_{\ell m}, B_{\ell m}$ condition is continuity of V at $r=R$. Thus, together with the orthogonality of the spherical harmonics, implies

$$A_{\ell m} R^{\ell} = B_{\ell m} R^{-(\ell+1)}$$

$$\rightarrow \quad A_{\ell m} R^{2\ell+1} = B_{\ell m}$$

The more subtle condition is Gauss' law (page 2-1)

$$r<R \qquad E_r = \frac{-\partial V}{\partial r} = -\sum_{\ell m} A_{\ell m}\, \ell\, r^{\ell-1}\, Y_{\ell m}(\theta,\phi) \qquad (\ell \neq 0)$$

$$r>R \qquad = \sum_{\ell m} B_{\ell m} (\ell+1) r^{-(\ell+2)} Y_{\ell m}(\theta,\phi)$$

whence from page 2-1 we see at $r=R$

$$* \quad \sum_{\ell m} \left[-A_{\ell m}\, \ell\, R^{\ell-1} + B_{\ell m} (\ell+1) R^{-(\ell+2)} \right] Y_{\ell m}(\theta,\phi) = \frac{\sigma_0}{\epsilon_0} \sin 2\theta \sin\phi$$

Looking up the forms of $Y_{\ell m}$ we notice

$$Y_{2,1}(\theta,\phi) = -\sqrt{\frac{5}{24\pi}}\, 3\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_{2,-1}(\theta,\phi) = +\sqrt{5/24\pi}\, 3\sin\theta\cos\theta\, e^{-i\phi}$$

But $\quad \sin 2\theta = 2\sin\theta\cos\theta \qquad \sin\phi = 1/2i\,(e^{i\phi} - e^{-i\phi})$

So that
$$\sin 2\theta \sin\phi = -\sqrt{\frac{24\pi}{5}}\,\frac{2}{3}\,\frac{1}{2i}\left(Y_{2,1} + Y_{2,-1}\right)$$

$$= \frac{i}{3}\sqrt{\frac{24\pi}{5}}\left(Y_{2,1} + Y_{2,-1}\right)$$

The orthogonality of the $Y_{\ell m}$ allows us to get $*$ on page 2-2 term by term. If $(\ell,m) \neq (2,\pm 1)$

$$-A_{\ell m}\,\ell R^{\ell-1} + B_{\ell m}(\ell+1) R^{-(\ell+2)} = 0$$

$$A_{\ell m}\,\frac{\ell}{\ell+1}\,R^{2\ell+1} = B_{\ell m}$$

But the continuity of V at $r=R$ gave the same eqn without the $\ell/\ell+1$ factor. The only way these can both be true is if $A_{\ell m} = B_{\ell m} = 0$. Our intuition might have suggested this... if the "source" the surface charge, only involves Y_{21} and $Y_{2,-1}$ the "result", induced potential also only involves these.

2-4

For $(\ell, m) = (2, 1)$

$$A_{21} R^5 = B_{21}$$

$$-A_{21}\, 2R + B_{21}\, 3R^{-4} = \frac{\sigma_0}{\epsilon_0} \frac{i}{3} \sqrt{\frac{24\pi}{5}}$$

$$-A_{21}\, 2R + A_{21}\, 3R = \sigma_0/\epsilon_0 \, \frac{i}{3} \sqrt{\frac{24\pi}{5}} = A_{21} R$$

$$= B_{21}/R^4$$

and the same for $(\ell, m) = (2, -1)$.

Thus outside the sphere $r > R$

$\underline{\underline{r > R}}$

$$V(r, \theta, \phi) = \frac{\sigma_0 R^4}{\epsilon_0} \frac{i}{3} \sqrt{\frac{24\pi}{5}} \left\{ Y_{2,1} + Y_{2,-1} \right\}$$

$$= \frac{\sigma_0 R^4}{\epsilon_0} \left\{ -i \sin\theta \cos\theta\, e^{i\phi} + i \sin\theta \cos\theta\, e^{-i\phi} \right\} r^{-3}$$

$$= \frac{\sigma_0 R^2}{\epsilon_0} \frac{R^2}{r^3} \underbrace{\sin 2\theta \sin\phi}_{\text{Same angular structure as surface charge density}}$$

$\sigma_0 R^2$ has units of Q

If you like can compute total charge on sphere

$$Q_{tot} = \int_0^{2\pi} R\, d\phi \int_0^{\pi} R \sin\theta\, d\theta \; 2 \sin\theta \cos\theta \sin\phi = 0$$

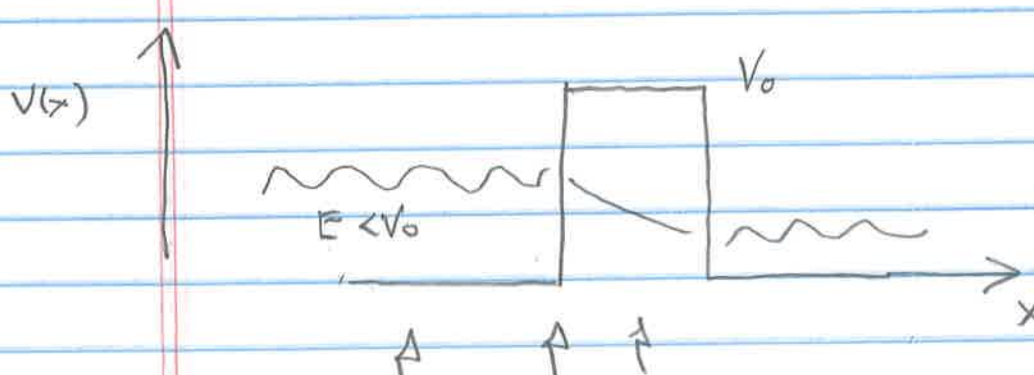
2-5

The fact that $Q_{top} = 0$ is why there is no $1/r$ term in $V(r,\theta,\phi)$ outside sphere.

Obviously the same substitutions can give $V(r,\theta,\phi)$ for $r<R$ and we get

$$\underline{\underline{r<R}} \qquad V(r,\theta,\phi) = \frac{\sigma_0 R^2}{\epsilon_0}\,\frac{r^2}{R^3}\,\sin 2\theta \sin\phi ,$$

Note the similarities of this part of problem with QM problems, eg a particle incident on a potential barrier



Solns here $e^{\pm ikx}$

Solns here $e^{-\kappa x}$

must match continuity of $\psi(x)$ and $d\psi/dx$ at bdy. This determines coefficients.

3-1

The general soln of Laplace's eqn in spherical coordinates obeying azimuthal symmetry, as in problem 1, is

$$V(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta)$$

In contrast to problem 1, since we are confined to $a < r < b$ we cannot let $r \to \infty$ (or $r \to 0$) so we cannot conclude $A_l = 0$ (or $B_l = 0$).

Fortunately, we have two functions $V_a(\theta)$ and $V_b(\theta)$ whose information is sufficient.

$$V_a(\theta) \equiv V(a,\theta) = \sum_{l=0}^{\infty} \left[A_l a^l + B_l a^{-(l+1)}\right] P_l(\cos\theta)$$

$$V_b(\theta) \equiv V(b,\theta) = \sum_{l=0}^{\infty} \left[A_l b^l + B_l b^{-(l+1)}\right] P_l(\cos\theta)$$

From the orthogonality of the Legendre polynomials

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = 2\delta_{nm}/(2n+1)$$

or, in terms of $x = \cos\theta$

$$\int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta) \sin\theta\, d\theta = 2\delta_{nm}/(2n+1)$$

If we multiply $*$ by $P_n(\cos\theta)\sin\theta$ and integrate,

3-2

$$\int_0^\pi V_a(\theta) P_n(\cos\theta)\sin\theta\, d\theta = \left(A_n a^n + B_n a^{-(n+1)}\right)\frac{2}{2n+1}$$

$$\int_0^\pi V_b(\theta) P_n(\cos\theta)\sin\theta\, d\theta = \left[A_n b^n + B_n b^{-(n+1)}\right] 2/_{2n+1}$$

Calling the integrals on the left hand side I_{an} and I_{bn} for short

$$* \qquad \frac{2n+1}{2} I_{an} = A_n a^n + B_n / a^{n+1}$$

$$** \qquad \frac{2n+1}{2} I_{bn} = A_n b^n + B_n / b^{n+1}$$

If we multiply $*$ by a^{n+1} and $**$ by b^{n+1} and subtract

$$\frac{2n+1}{2} I_{an} a^{n+1} - \frac{2n+1}{2} I_{bn} b^{n+1} = A_n \left(a^{2n+1} - b^{2n+1}\right)$$

whence

$$A_n = \frac{2n+1}{2}\left(\frac{1}{a^{2n+1} - b^{2n+1}}\right)\left(I_{an} a^{n+1} - I_{bn} b^{n+1}\right)$$

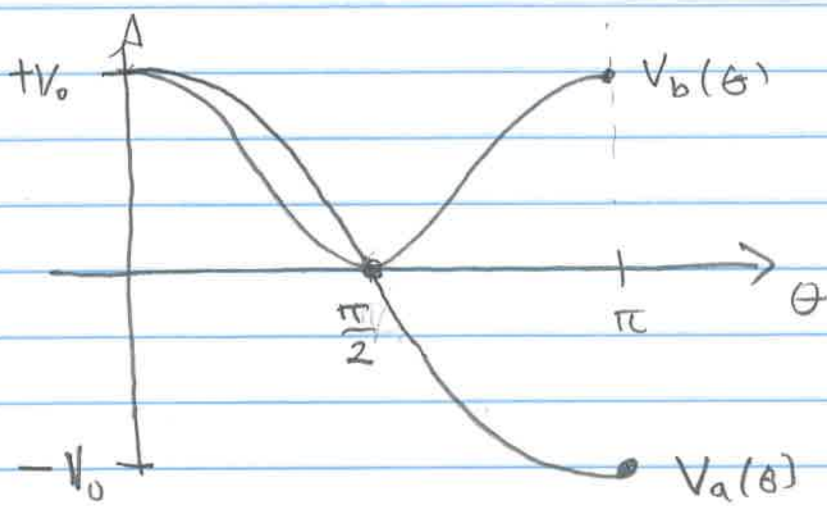
There is a similar expression for B_n:

$$B_n = \frac{2n+1}{2}\left(\frac{b^{2n+1} a^{2n+1}}{a^{2n+1} - b^{2n+1}}\right)\left(I_{an}/a^n - I_{bn}/b^n\right)$$

3-3

You are asked to do an "amusing" case.

How about $V_a(\theta) = V_0 \cos\theta$ $V_b(\theta) = V_0 \cos^2\theta$



Clearly $V_a(\theta) = P_1(\cos\theta)$ so that $I_{a(n=1)} = \frac{2}{3} V_0$

with all other $I_{an} = 0$. Likewise $I_{b(n=1)} = 2/3\, V_0$

and $I_{b(n=2)} = 2/3\; 2/5\, V_0$ with all other $I_{bn} = 0$ because

$V_b(\theta) = 2/3\; P_2(\cos\theta) + P_1(\cos\theta)$

We conclude all A_n and B_n vanish except for $n=1$ and $n=2$ and, as an example of a nonvanishing term

$$A_1 = \frac{3}{2}\left(\frac{1}{a^3-b^3}\right)\left(\frac{2}{3}a^2 - \frac{2}{3}b^2\right)V_0$$

$$= \frac{a^2-b^2}{a^3-b^3} V_0 .$$

similar expressions for A_2, B_1 B_2 ...