PHYSICS 200B, WINTER 2017
ELECTRICITY AND MAGNETISM

Assignment Two, Due Friday, February 3, 5:00 pm.

[0.] Problem Six from Assignment One.

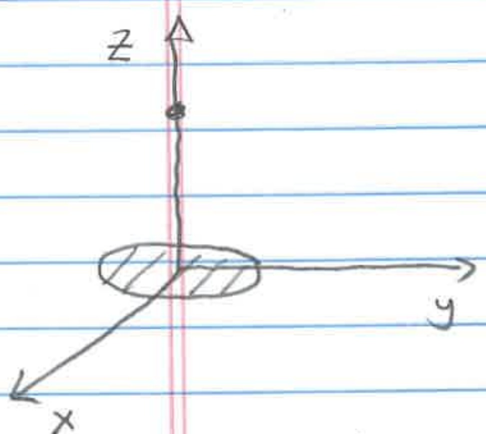
[1.] Compute the potential $V(r, \theta, \phi)$ due to a thin disk of charge Q of constant charge per area. Make a convenient choice of origin and orientation of your axes.

[2.] A point charge Q is a distance z_0 from an infinite metallic conducting plane held at potential $V = 0$. Compute the induced charge density on the surface of the plane and the total charge on the plane. What is the electric field (magnitude and direction) at the surface of the plane?

[3.] A point charge Q is a distance z_0 from the center of a metallic sphere of radius $R < z_0$ held at a potential $V = 0$. What is the Greens function for positions outside the sphere? What is the potential outside the sphere?

Physics 200B PS2 Solutions W 2017

We first compute the potential along a high symmetry direction – the z axis. We do this by integrating the potential due to a collection of rings of charge which together build up the disk



$$V_{ring} = \frac{2\pi\rho\, d\rho\, \sigma}{4\pi\epsilon_0\sqrt{\rho^2+z^2}}$$

← charge on ring of radius ρ

↖ distance of charge to point on z axis

disk radius

$$V_{disk} = \int_0^a d\rho\; V_{ring}$$

$$= \frac{\sigma}{2\epsilon_0}\int_0^a \rho\, d\rho \Big/ \sqrt{\rho^2+z^2}$$

$$= \frac{\sigma}{2\epsilon_0}\left(\rho^2+z^2\right)^{1/2}\Big|_0^a$$

$$= \frac{\sigma}{2\epsilon_0}\left\{\left(a^2+z^2\right)^{1/2} - z\right\}$$

Check this by considering $z \gg a$

$$\left(a^2+z^2\right)^{1/2} = z\left(1+a^2/z^2\right)^{1/2} \simeq z\left(1+a^2/2z^2\right)$$

Thus for $z \gg a$

$$V_{disk} \rightarrow \frac{\sigma}{2\epsilon_0}\left\{ z + a^2/2z - z \right\} = \frac{\sigma a^2}{4z\epsilon_0}$$

using $Q = \pi a^2 \sigma$ $\quad V_{disk} = \frac{Q}{4\pi\epsilon_0 z}$

which makes sense: at large distances disk appears as a point charge.

Meanwhile for $z \ll a$

$$V_{disk} \rightarrow \frac{\sigma}{2\epsilon_0}\left\{ a + z^2/2a - z \right\} \rightarrow V_0 - \frac{\sigma}{2\epsilon_0} z$$

This also makes sense! The field E near the disk is that of an infinite plane $E = \sigma/2\epsilon_0$. Since $E_z = -dV/dz$ we must have $V = -\sigma z/2\epsilon_0$.

The second part of the problem leverages this high symmetry soln to the entire upper half plane, ie off the z-axis. By symmetry (azimuthal) V will depend only on r and θ and we know

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left[a_\ell r^\ell + b_\ell r^{-(\ell+1)}\right] P_\ell(\cos\theta)$$

z axis is $\theta = 0$ and $P_\ell(1) = 1$

1-3

$$\sum_{\ell=0}^{\infty}\left[a_\ell r^\ell + b_\ell r^{-(\ell+1)}\right] 1 = \frac{\sigma}{2\epsilon_0}\left\{(a^2+r^2)^{1/2} - r\right\}$$

$z \to r$ for $\theta = 0$

Assuming we are interested in $r > a$, right hand side is

$$\frac{\sigma}{2\epsilon_0}\left\{r\left(1+a^2/r^2\right)^{1/2} - r\right\}$$

$$= \frac{\sigma}{2\epsilon_0}\left\{r\left[1 + \frac{a^2}{2r^2} - \frac{1}{8}\frac{a^4}{r^4} + \frac{1}{16}\frac{a^6}{r^6}\ \ldots\right] - r\right\}$$

$$\frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) \qquad \frac{1}{2}\frac{1}{3}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)$$

$$= \frac{\sigma}{2\epsilon_0}\left\{a^2/2r - 1/8\ a^4/r^3 + a^6/16r^5\ \ldots\right\}$$

We conclude that $a_\ell = 0\ \forall \ell$ and $b_\ell = 0$ for ℓ odd

and specifically $b_0 = \frac{\sigma a^2}{4\epsilon_0} = Q/4\pi\epsilon_0$

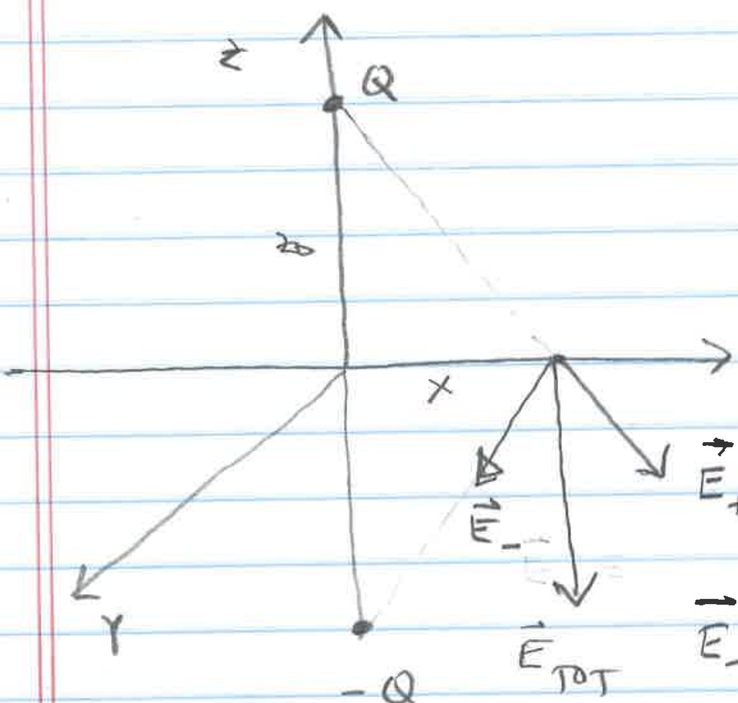
$$b_3 = -\frac{\sigma a^4}{16\epsilon_0} = -\frac{Q a^2}{16\pi\epsilon_0}$$

$$b_5 = +\frac{\sigma a^6}{32\epsilon_0} = +\frac{Q a^4}{32\pi\epsilon_0} \quad \text{etc.} \ldots$$

2-1

Can satisfy bdy condition that $V=0$ on conducting ~~plane~~

by placing $-Q$ image charge on opposite side



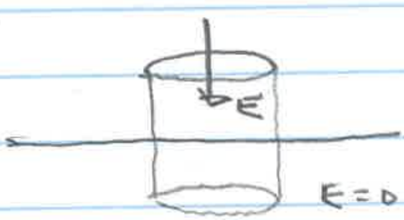
$$\vec{E}_+ = \frac{Q}{4\pi\epsilon_0}\frac{1}{x^2+z_0^2}\frac{x\hat{i}-z_0\hat{k}}{(x^2+z_0^2)^{1/2}}$$

$$\vec{E}_- = \frac{Q}{4\pi\epsilon_0}\frac{1}{x^2+z_0^2}\frac{-x\hat{i}+z_0\hat{k}}{(x^2+z_0^2)^{1/2}}$$

$$\vec{E}_{TOT} = -Qz_0/2\pi\epsilon_0(x^2+z_0^2)^{3/2}$$

in general ρ if not on x axis

charge density on plane related to $|\vec{E}|$ by Gauss' law



$$\Phi_E = |E|A = \frac{Q}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$\Rightarrow \quad \sigma = \epsilon_0|E|$$

Here $\sigma < 0$ clearly, to get $E_z < 0$

Integrate to get total charge Q_p on plane

$$Q_p = \int_0^\infty \underbrace{2\pi \rho \, d\rho}_{dA} \, \sigma = \int_0^\infty 2\pi \rho \, d\rho \, \epsilon_0 \left\{ - \frac{Q z_0}{2\pi\epsilon_0 (\rho^2 + z_0^2)^{3/2}} \right\}$$

$$= - Q z_0 \int_0^\infty \rho \, d\rho \, 1/(\rho^2 + z_0^2)^{3/2}$$

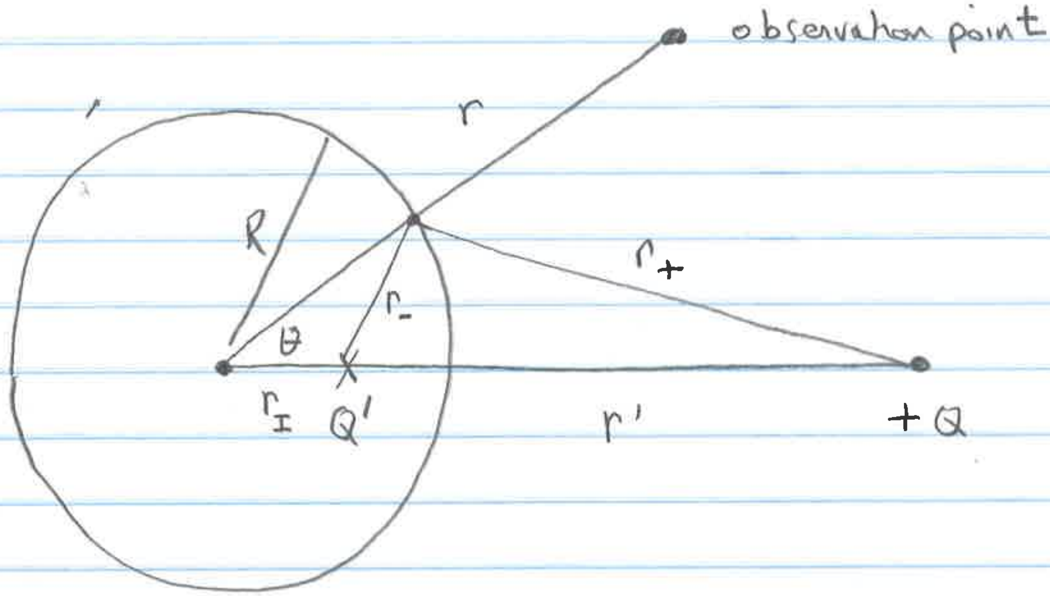
$$= - Q z_0 \left. -1/(\rho^2 + z_0^2)^{1/2} \right|_0^\infty$$

$$= - Q z_0 \left\{ 0 + 1/z_0 \right\} = - Q$$

$Q_p = -Q$ Exactly matches image charge

3-1

It is probably more consistent with our visual notation to denote the position of the point charge as r' rather than z_0



"image charge"

We want to find the location r_I and magnitude Q' which will give $V=0$ on the surface of the sphere

Clearly $Q' < 0$ and also its location, by symmetry, must be on line connecting sphere center and charge $+Q$.

$$0 = V_R = \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{r_+} - \frac{\alpha}{r_-}\right\} \qquad \alpha \equiv Q'/Q$$

$$r_+^2 = R^2 + r'^2 - 2Rr'\cos\theta$$

$$r_-^2 = R^2 + r_I^2 - 2Rr_I\cos\theta$$

To get this satisfied $\quad 1/r_+ = \alpha/r_-$

$$r_-^2 = \alpha^2 r_+^2$$

$$R^2 + r_I^2 - 2Rr_I\cos\theta = \alpha^2\left(R^2 + r'^2 - 2Rr'\cos\theta\right)$$

This is true if $\quad R^2 + r_I^2 = \alpha^2(R^2 + r'^2)$

$$2Rr_I = \alpha^2 2Rr'$$

The bottom eqn gives $\quad r_I = \alpha^2 r'$ and then

$$R^2 + r_I^2 = \frac{r_I}{r'}(R^2 + r'^2)$$

$$r_I^2 - \frac{R^2 + r'^2}{r'} r_I + R^2 = 0$$

$$r_I = \frac{1}{2}\left\{\frac{R^2 + r'^2}{r'} \pm \sqrt{\left(\frac{R^2 + r'^2}{r'}\right)^2 - 4R^2}\right\}$$

$$\frac{1}{r'^2}\left\{R^4 + 2R^2r'^2 + r'^4 - 4R^2r'^2\right\}$$

$$= \frac{1}{r'^2}(R^2 - r'^2)^2$$

$$r_I = \frac{1}{2}\frac{1}{r'}\left\{(R^2 + r'^2) \pm (R^2 - r'^2)\right\}$$

The $-$ sol'n gives $r_\pm = r'$ and then $\alpha = 1$

This is an (expected) trivial sol'n. One can get $V=0$

on the sphere surface by placing $-Q$ $(\alpha=1)$ right

on top $(r_I = r')$ of $+Q$. In fact $V=0$ everywhere!

The $+$ soln is not trivial, giving

$r_\pm = R^2/r'$ $\leftarrow$ $r_I < R$ since $R/r' < 1$

$\alpha = R/r'$ $\leftarrow$ $\alpha < 1$ since $R/r' < 1$

Then the greens function is

$$G(r,r') = \frac{1}{|r-r'|} - \frac{R/r'}{|r - (R^2/r'^2)\, r'|}$$

To get the potential revert to notation in problem statement

and place $+Q$ at $\vec{z_0}$

$$V(\vec{r}) = \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{|\vec{r}-\vec{z_0}|} - \frac{R/z_0}{|\vec{r} - R^2/z_0^2\, \vec{z_0}|}\right\}$$