PHYSICS 200C, SPRING 2016
ELECTRICITY AND MAGNETISM

Assignment One, Due Friday, January 20, 5:00 pm.

[1.] A crude model of the H_2 molecule is that the electrons form a spherical cloud of charge of radius a and the two protons are point charges inside this sphere. Find the equilibrium proton positions.

[2.] In certain liquids, when a point impurity q is placed at the origin, the electrostatic potential is,

$$\phi(r) = \frac{q}{4\pi\epsilon_0 r} e^{-r/\lambda} .$$

λ is called the Debye-Huckel length. Find the electric field and charge density everywhere, and the total charge inside an infinitely large sphere.

[3.] A hydrogen atom acts as if it had an electrostatic potential

$$\phi(r) = \frac{q}{4\pi\epsilon_0 r}(1 + \frac{r}{a_0})e^{-2r/a_0} ,$$

where q is the charge on the proton and $a_0 = \hbar^2/m_e q^2 = 0.529\dot{A}$ is the Bohr radius.
(a) Find the corresponding charge density and interpret the various terms physically.
(b) Verify by *explicit* integration that your resulting charge density from part (a) indeed produces the original potential.
(c) What is the net charge inside a sphere of radius a_0? What is the electric field strength at this distance?

[4.] A sphere of radius a has a charge Q uniformly distributed thhroughout its volume.
(a) Obtain the total electrostatic energy $3Q^2/(20\pi\epsilon_0 a)$ directly from the expression,

$$U = \frac{1}{8\pi\epsilon_0}\int d^3r \int d^3r' \, \frac{\rho(\vec{r}\,)\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} .$$

(b) Use Gauss' Law to find the electric field throughout all space. Rederive the total energy by integrating $\frac{1}{2}\epsilon_0 E^2$.
(c) Suppose the same charge is instead spread uniformly over the *surface* of the sphere. Use either of the above methods to find the corresponding total energy, and discuss *physically* why and how it differs from the uniformly charged sphere.

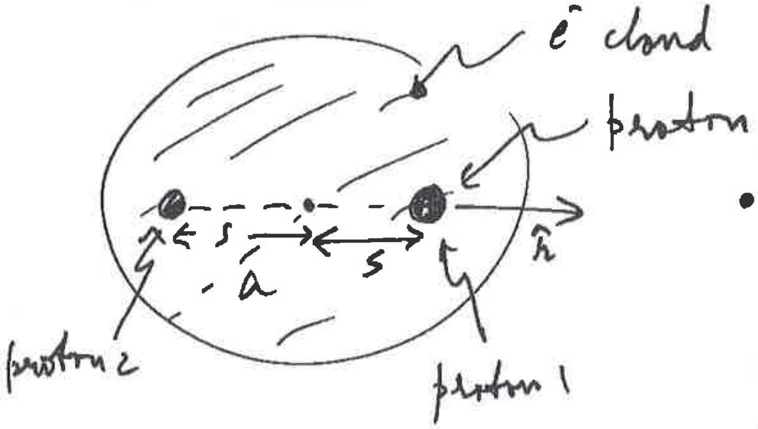
[5.]
(a) Suppose $G(\vec{r}, \vec{r}\,')$ satisfies the inhomogeneous equation $\nabla_r^2 G(\vec{r}, \vec{r}\,') = -\delta(\vec{r} - \vec{r}\,')/\epsilon_0$, subject to the boundary condition $G(\vec{r}, \vec{r}\,') = 0$ for $\vec{r}$ on all surrounding surfaces $\mathcal{S}$. Use Green's theorem to solve the equation $\nabla_r^2 \phi(\vec{r}\,) = -\rho(\vec{r}\,)/\epsilon_0$, with the specified boundary conditions $\phi(\vec{r}\,) = -f(\vec{r}\,)$ on $\mathcal{S}$.
(b) As an example, consider the empty half-space $z > 0$. The boundary plane at $z = 0$ is held at zero potential except for an equilateral triangle of side a at a potential V. Find the appropriate $G(\vec{r}, \vec{r}\,')$ and hence obtain an integral expression for the potential throughout the region $z > 0$. Evaluate the first nonvanishing term in $\phi(\vec{r}\,)$ as $|\vec{r}\,| \to \infty$.

[5.] (continued)
(c) A point charge q is now added to the configuration in part (b). The charge is placed a distance z_0 from the plane along a line through the center of the triangle and normal to its plane. Find an integral representation for the total potential and evaluate the first nonvanishing contribution for $|\vec{r}| \to \infty$. Identify and interpret the various terms.

[6.] A conducting sphere of radius a has a total charge Q. Of this charge, $Q - Q_E$ is distributed uniformly through the volume of the sphere, while the remaining Q_E is uniformly distributed in a ring of zero thickness around the equator of the surface of the sphere. Assume that Q and Q_E are of the same sign.
(a) By direct integration, find the potential $\phi(\vec{r})$ along the polar axis of the sphere for $|\vec{r}| > a$.
(b) By expressing the potential in a power series in spherical coordinates, and comparing with your answer in (a), determine $\phi(\vec{r})$ for all points $\vec{r}$ such that $|\vec{r}| > a$.
(c) Calculate the electric field for large $|\vec{r}|$ to order $a^2/|\vec{r}|^2$. Then consider a particle of charge q (q and Q are of opposite signs) which is held by electrostatic attraction in a bound orbit around the sphere. Show that such an orbit is stable in *inclination* (the angle between the orbit plane and the equatorial plane of the sphere) only if the inclination is zero. Discuss the gravitational analog of this problem. (Recall the earth has an equatorial bulge.)

A//

(1) A very crude model of the H_2 molecule is as follows: The electron cloud is a sphere of charge of radius a, and the two protons are point charges at rest, embedded inside this sphere.
Find a configuration for the protons which is in equilibrium, and find how far the protons are from the center.



• Let one proton be at a distance s. The $\vec{E}$-field from the e^- cloud at its location is $-\frac{2e}{a^3} s \hat{r}$ in the Gaussian system.

(How do we know this? We know (from problem on Thompson's model of atom) that $|\vec{E}|$ rises linearly with distance inside the cloud, and it must have a value $2e/a^2$ at the surface.) (charge of 2 electrons)

The second proton must provide a cancelling force. In order that the force on it (proton 2) is also zero, it must be at the diametrically opposite point a distance s from the center. So we must have

$$-\frac{2e^2}{a^3} s + \frac{e^2}{(2s)^2} = 0, \quad \text{i.e.} \quad (2s)^3 = a^3$$

$$\text{or} \quad s = \frac{a}{2}.$$

Comment (unrelated to question asked): The equilibrium is stable, but only marginally. It is not unstable.

B//

2\. } In certain liquids, when a point impurity with charge q is placed at $r = 0$, the electrostatic potential in the liquid is (in SI)

$$\phi(r) = \frac{q}{4\pi\epsilon_0 r} e^{-r/\lambda},$$

where λ is a constant known as the Debye-Huckel length. Find the electric field and charge density everywhere, and the total charge inside an infinitely large sphere.

$$\vec{E} = -\vec{\nabla}\phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r^2} + \frac{1}{r\lambda}\right) e^{-r/\lambda} \, \hat{r}.$$

In SI, $\nabla^2\phi = -\rho/\epsilon_0$, so

$$\rho(r) = -\nabla^2\left[\frac{q}{4\pi r} e^{-r/\lambda}\right]$$

Recall the Laplacian of the $\frac{1}{r}f(r)$ form.

$$= q\,\delta(\vec{r})\left(e^{-r/\lambda}\right)_{r=0} - \frac{q}{4\pi r}\frac{d^2}{dr^2}\left(e^{-r/\lambda}\right).$$

$$= q\,\delta(\vec{r}) - \frac{q}{4\pi\lambda^2 r} e^{-r/\lambda}.$$

$$\text{Total charge} = Q_{tot} = \int \rho(r)\, d^3r = q - \frac{q}{4\pi\lambda^2}\int_0^\infty \frac{e^{-r/\lambda}}{r}\cdot 4\pi r^2\, dr$$

$$= q - \frac{q}{\lambda^2}\underbrace{\int_0^\infty r\, e^{-r/\lambda}\, dr}_{\lambda^2\, 1! = \lambda^2}$$

$$= q - q = 0.$$

Comment: The second term in $\rho(r)$ arises from differential movement of + & − ions toward the impurity, and is known as the screening charge.

1.

Problem Set #1 P200B PS 1 Richard Scalettar

NB: Sol'ns are in CGS units.

$\boxed{3.}$ Hydrogen atom has potential $\Phi(r) = q/r\,(1+r/a_0)\,e^{-2r/a_0}$ where q is charge on proton and $a_0 = \hbar^2/m_e q^2 \approx 0.529\,\text{Å}$ is Bohr radius.

a) Find $\rho(r)$ and interpret physically.

We know that $\nabla^2 \Phi(r) = -4\pi\rho(r)$. In a case like this, there is no θ and φ dependence and the Laplacian is $1/r\, d^2/dr^2\, r$

Thus for $r \neq 0$

$$\rho_e(r) = -\frac{1}{4\pi}\frac{1}{r}\frac{d^2}{dr^2}\, q\,(1+r/a_0)\,e^{-2r/a_0}$$

$$= -q/4\pi r \frac{d}{dr}\left(1/a_0\, e^{-2r/a_0} + -2/a_0\,(1+r/a_0)\,e^{-2r/a_0}\right)$$

$$= -q/4\pi r\left(\cancel{-2/a_0^2\, e^{-2r/a_0}} + 4/a_0^2\,(\cancel{1}+r/a_0)\,e^{-2r/a_0} - \cancel{2/a_0^2\, e^{-2r/a_0}}\right)$$

$$\rho_e(r) = -\frac{q}{\pi a_0^3}\, e^{-2r/a_0} \quad \text{for } r \neq 0$$

when $r=0$ we must recognize $\nabla^2 \frac{1}{r} = -4\pi\,\delta(\vec{r})$ so that we get an extra term

$$\rho_p(r) = -\frac{1}{4\pi} - 4\pi q\,\delta(\vec{r}) = q\,\delta(\vec{r})$$

All together $$\rho(\vec{r}) = q\,\delta(\vec{r}) - q\,\frac{1}{\pi a_0^3}\, e^{-2r/a_0}$$

The physical interpretation is clear: The first term is the charge density of the proton which is localized exactly at $r=0$. The second term is the charge density of the electron "cloud" around the proton. In fact, the ground state wave function is $\psi_0(r) = 1/\sqrt{4\pi}\; 2/a_0^{3/2}\, e^{-r/a_0}$ (Baym) so that $|\psi_0(r)|^2 = \frac{1}{4\pi}\frac{4}{a_0^3}e^{-2r/a_0} = 1/\pi a_0^3\, e^{-2r/a_0}$. Thus the second term is simply $-q\,|\psi_0(r)|^2$ as expected.

We might also comment that we expect the atom to be electrically neutral overall. So we verify that

$$Q_{TOT} = \int \rho(\vec r)\,dV = q + \int_0^\infty 4\pi r^2 \frac{-q}{\pi a_0^3} e^{-2r/a_0}\, dr$$

$$= q\left(1 - \frac{4}{a_0^3}\int_0^\infty r^2 e^{-2r/a_0}\, dr\right)$$ let $u = +2r/a_0$ $\quad dr = a_0/2\, du$

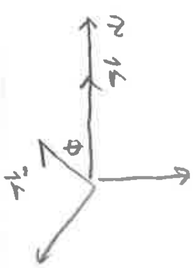
$$= q\left(1-\frac{4}{a_0^3}\left(\frac{a_0}{2}\right)^3\int_0^\infty u^2 e^{-u}\, du\right)$$ But $\int_0^\infty u^n e^{-u}\, du = n!$

$$= q\left(1-\frac{1}{2}\, 2!\right) = 0$$

Thus $Q_{TOT} = 0$
The total charge density integrates to zero – the H atom is electrically neutral.

(b) Verify by explicit integration that $\rho(r)$ reproduces $\Phi(r)$.

$$\Phi(\vec r) = \int \frac{\rho(\vec r')\, d\vec r'}{|\vec r - \vec r'|}$$



Choose for convenience $\vec r$ to lie along $\hat z$-axis. Then
$|\vec r-\vec r'|^2 = r^2 - 2rr'\cos\theta + r'^2$ and hence

$$\Phi(r) = q\int \frac{\delta(\vec r') - 1/\pi a_0^3\, e^{-2r'/a_0}}{\sqrt{r^2-2rr'\cos\theta'+r'^2}}\, d\vec r'$$ where $\delta(\vec r') = \delta(r')\delta(\theta')\delta(\phi')\frac{1}{r'^2\sin\theta'}$

$d\vec r' = r'^2 \sin\theta'\, d\theta'\, d\phi'\, dr'$

$$= q\int_0^\infty r'^2 dr' \int_0^{2\pi} d\phi' \int_0^\pi \sin\theta'\, d\theta' \left\{\frac{\delta(r')\delta(\theta')\delta(\phi')}{r'^2\sin\theta'} - \frac{1}{\pi a_0^3}e^{-2r'/a_0}\right\}\frac{1}{\sqrt{r'^2-2rr'\cos\theta'+r^2}}$$

$$= q/r - \frac{2\pi q}{\pi a_0^3}\int_0^\infty r'^2\, dr'\, \frac{1}{rr'}\sqrt{r^2-2rr'\cos\theta + r'^2}\,\Big|_0^\pi\, e^{-2r'/a_0}$$

$$\Phi(r) = q/r\left\{1 - \frac{2}{a_0^3}\int_0^\infty dr'\; r'\left(|r+r'| - |r-r'|\right)e^{-2r'/a_0}\right\}$$

$$= q/r\left\{1 - 2/a_0^3\left[\int_0^r dr'\; r'\, 2r'\, e^{-2r'/a_0} + \int_r^\infty dr'\; r'\; 2r\, e^{-2r'/a_0}\right]\right\}$$

 Scalettar

Now $\int_0^r r' e^{-2r'/a_0} dr' = -r' a_0/2\, e^{-2r'/a_0}\Big|_0^r + a_0/2 \int_0^r e^{-2r'/a_0} dr'$

$= -r\, a_0/2\, e^{-2r/a_0} + a_0/2 (-a_0/2) e^{-2r'/a_0}\Big|_0^r$

$= -r\, a_0/2\, e^{-2r/a_0} + a_0^2/4 - a_0^2/4\, e^{-2r/a_0}$

and so $\int_0^r r'^2 e^{-2r'/a_0} dr' = + r'^2 \frac{a_0}{2} e^{-2r'/a_0}\Big|_0^r + \frac{a_0}{2}\, 2 \int_0^r r' e^{-2r'/a_0} dr'$

$= -a_0 r^2/2\, e^{-2r/a_0} + 4/a_0 \left(-r\, a_0/2\, e^{-2r/a_0} + a_0^2/4 - a_0^2/4\, e^{-2r/a_0}\right)$

Also $\int_r^\infty dr'\, r' e^{-2r'/a_0} = -\frac{a_0}{2} r' e^{-2r'/a_0}\Big|_r^\infty + a_0/2 \int_r^\infty e^{-2r'/a_0}$

$= \frac{a_0}{2} r e^{-2r/a_0} - a_0^2/4\, e^{-2r'/a_0}\Big|_r^\infty = \frac{a_0}{2} r e^{-2r/a_0} + \frac{a_0^2}{4} e^{-2r/a_0}$

Thus, all together

$\phi(r) = q/r \{ 1 - \frac{2}{a_0^3}\left[-a_0 r^2 - a_0^2 r - a_0^3/2 + a_0 r^2 + a_0^2/2\, r\right] e^{-2r/a_0}$

$- 2/a_0^3 [\, a_0^3/2 \,] \}$

$= q/r\, (-2/a_0^3)(-\frac{a_0^2}{2} r - a_0^3/2)\, e^{-2r/a_0}$

$$\boxed{\phi(r) = q/r\, (1 + r/a_0)\, e^{-2r/a_0}}$$

as required.

(c) calculate net charge in sphere of radius a_0

$Q(a_0) = \int_0^{a_0} 4\pi r^2 dr\; q\left(\frac{\delta(r)}{4\pi r^2} - \frac{1}{\pi a_0^3} e^{-2r/a_0}\right)$

$= q - \frac{4q}{a_0^3} \int_0^{a_0} r^2 e^{-2r/a_0} dr = q - \frac{4q}{a_0^3}\left\{ -r^2 \frac{a_0}{2} e^{-2r/a_0}\Big|_0^{a_0} + a_0 \int_0^{a_0} r e^{-2r/a_0} dr \right\}$

$= q\left(1 - \frac{4}{a_0^3}\left(-\frac{a_0^3}{2} e^{-2} + \frac{a_0^2}{2} r e^{-2r/a_0}\Big|_0^{a_0} + \frac{a_0^2}{2}\int_0^{a_0} e^{-2r/a_0} dr\right)\right)$

$$Q(a_0) = q\left(1 - \frac{4}{a_0^3}\left(-\frac{a_0^3}{2e^2} - \frac{a_0^3}{2e^2} + \frac{a_0^3}{4}\, e^{-2r/a_0}\Big|_0^{a_0}\right)\right)$$

$$= q\left(1 - \frac{4}{a_0^3}\left(-a_0^3/e^2 + a_0^3/4e^2 + a_0^3/4\right)\right)$$

$$= q\left(1 + \frac{4}{e^2} + \frac{1}{e^2} + 1\right) = 5q/e^2$$

$$\boxed{Q(a_0) = \frac{5}{e^2}q = .677\, q}$$

One way to look at this is to say a sphere of radius a_0 contains the proton charge $+q$ and $(1 - 5/e^2) = 32.3\ \%$ of the total electron charge $-q$.

The electric field strength at a_0 is given by Gauss' law

$$E(a_0)\, \cancel{4\pi} a_0^2 = \cancel{4\pi}\, Q(a_0)$$

$$E(a_0) = \frac{5}{e^2}\, q/a_0^2$$

In MKS units $E(a_0) = \dfrac{5}{e^2}\, \dfrac{1.609 \cdot 10^{-19}}{(.529 \cdot 10^{-10})^2}\, 9 \cdot 10^9 = 3.5 \cdot 10^{11}\ \dfrac{\text{volts}}{\text{m}}$

$$\boxed{E(a_0) = 3.5 \cdot 10^9\ \frac{\text{volts}}{\text{cm}}}$$

According to Jackson p820 $1\ \text{volt/m} = \frac{1}{3} \cdot 10^{-4}\ \dfrac{\text{statvolt}}{\text{cm}}$

so that $\boxed{E(a_0) = 1.17 \cdot 10^7\ \dfrac{\text{statvolt}}{\text{cm}}}$

This can also be computed directly:

$$E(a_0) = \frac{5}{e^2}\, q/a_0^2 = \frac{5}{e^2}\, \frac{4.8 \cdot 10^{-10}\ \text{statcoul.}}{(.529 \cdot 10^{-8}\,\text{cm})^2} = 1.17 \cdot 10^7\ \frac{\text{statvolt}}{\text{cm}}$$

4. Sphere radius a with charge Q distributed uniformly inside

a) show $U = 3Q^2/5a$

$$U = \tfrac{1}{2}\int d^3r \int d^3r' \; \rho(\vec r)\rho(\vec r')/|\vec r - \vec r'| \qquad \rho(\vec r) = 3Q/4\pi a^3$$

In doing $\int d^3r'$ pick axes so that $\vec r$ lies along $\hat z'$ axis.

$$U = \frac{1}{2}\, 9Q^2/16\pi^2 a^6 \int d^3r \int_0^a r'^2 dr' \int_0^{2\pi} d\phi' \int_0^\pi \sin\theta' d\theta' \, (r^2 - 2rr'\cos\theta' + r'^2)^{-1/2}$$

Notice integrand is indep of θ, ϕ, ϕ' so these integrals can be done immediately. In the θ' integration, change variables to $\mu \equiv \cos\theta'$

$$U = \frac{1}{2}\frac{9Q^2}{16\pi^2 a^6}\, 2\; 2\pi\; 2\pi \int_0^a r^2 dr \int_0^a r'^2 dr' \int_{-1}^1 d\mu \, (r^2 - 2rr'\mu + r'^2)^{-1/2}$$

Now $\int_{-1}^1 d\mu \, (r^2 - 2rr'\mu + r'^2)^{-1/2} = \frac{-1}{rr'}(r^2 - 2rr'\mu + r'^2)^{1/2}\Big|_{-1}^{1}$

$$= \frac{1}{rr'}\{|r+r'| - |r-r'|\} = \frac{1}{rr'}\begin{cases} 2r' & r'<r \\ 2r & r'>r \end{cases}$$ Therefore

$$U = \frac{9Q^2}{4a^6}\int_0^a r\,dr \left\{ \int_0^r r' dr'\, 2r' + \int_r^a r' dr'\, 2r \right\}$$

$$= 9Q^2/4a^6 \left\{ \int_0^a r\,dr \frac{2r^3}{3} + \int_0^a 2r^2 dr \left(\frac{a^2}{2} - \frac{r^2}{2}\right) \right\}$$

$$= 9Q^2/4a^6 \left\{ \frac{2}{3}\frac{a^5}{5} + a^2\frac{a^3}{3} - \frac{a^5}{5} \right\}$$

$$= 9Q^2/4a \left\{ \frac{2}{15} + \frac{1}{3} - \frac{1}{5} \right\} = 9Q^2/4a \;\; 4/15 = \boxed{3Q^2/5a = U}$$

(b) For $r>a$ Gauss' law becomes $E 4\pi r^2 = 4\pi Q$

For $r<a$ Gauss' law becomes $E 4\pi r^2 = 4\pi \frac{4}{3}\pi r^3 \, Q / \frac{4}{3}\pi a^3$

Thus $\boxed{E = \begin{cases} Q/r^2 & r>a \\ Qr/a^3 & r<a \end{cases}}$

$$U = \frac{1}{8\pi}\left\{\int_0^a 4\pi r^2 \frac{Q^2 r^2}{a^6}\,dr + \int_a^\infty 4\pi r^2 \frac{Q^2}{r^4}\,dr\right\}$$

$$= \frac{1}{2} Q^2 \left\{\frac{1}{5a} + \frac{1}{a}\right\} = \frac{1}{2}\frac{Q^2}{a}\frac{6}{5} = \boxed{3Q^2/5a = U.}$$

(c) Charge spread uniformly on surface of sphere. Calculate total energy and discuss physically the difference.

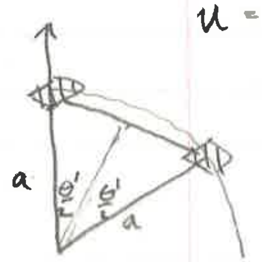
Since $E = \begin{cases} Q/r^2 & r>a \\ 0 & r<a \end{cases}$ we see from part (b) that $\boxed{U = \frac{Q^2}{2a}}$

we could also use the method of part (a)

$$U = \frac{1}{2}\int_0^\pi a^2 \sin\theta\, d\theta \int_0^{2\pi} d\phi \int_0^\pi a^2 \sin\theta'\, d\theta' \int_0^{2\pi} d\phi' \left(\frac{Q}{4\pi a^2}\right)^2 \frac{1}{2a\sin\theta'/2}$$

$$= \frac{1}{\cancel{2}}\cancel{a^4}\,\frac{Q^2}{16\cancel{\pi^2}\cancel{a^4}}\,\frac{1}{2a}\,\cancel{2\pi}\,2\cancel{\pi}\,\cancel{2}\int_0^\pi \frac{\sin\theta'\,d\theta'}{2\sin\theta'/2}$$

$$= \frac{Q^2}{8a}\int_0^\pi \frac{2\sin\theta'/2\,\cos\theta'/2}{\sin\theta'/2}\,d\theta' = \frac{Q^2}{4a}\, 2\sin\frac{\theta'}{2}\Big|_0^\pi = Q^2/2a$$



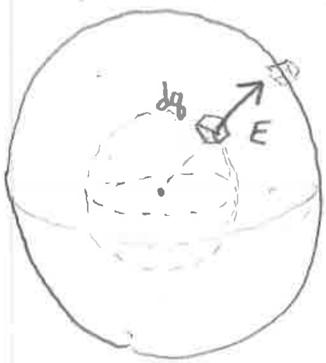
distance between two patches of charge = $2a\sin\theta'/2$

Thus method of part a yields $\boxed{Q^2/2a = U \text{ also}}$

Physically, the reason the energy is less for charge concentrated on surface is as follows: Imagine charge is uniformly distributed throughout sphere. Then we saw in part (b) that an electric field exists within the sphere pointing outward. Therefore to move charge from interior to surface releases energy. Thus it is clear that the energy of surface distribution should be less than uniform volume distribution, and, indeed

$$Q^2/2a < 3\,Q^2/5a$$

In fact, we can easily calculate this energy release



$$\Delta U = \int_0^a \underbrace{4\pi r^2 dr \frac{Q}{\frac{4}{3}\pi a^3}}_{dq} \left\{ \underbrace{\frac{Q\frac{4}{3}\pi r^3}{\frac{4}{3}\pi a^3}\left(\frac{1}{r}-\frac{1}{a}\right)}_{\Delta V} \right\}$$

$$\Delta U = \frac{3Q^2}{a^6}\int_0^a r^4 - \frac{r^5}{a}\, dr$$

$$= \frac{3Q^2}{a^6}\left(\frac{a^5}{5}-\frac{a^5}{6}\right) = \frac{3Q^2}{a}\left(\frac{1}{30}\right) = \frac{Q^2}{10a}$$

and, indeed, $\left(\frac{3}{5}-\frac{1}{2}\right)\frac{Q^2}{a} = \frac{1}{10}\frac{Q^2}{a}$.

$\boxed{5}$ a) Suppose $G(\bar r \bar r')$ satisfies $\nabla_r^2 G(\bar r \bar r') = -4\pi\,\delta(\bar r - \bar r')$

subject to $G(\bar r \bar r') = 0$ $\forall \bar r$ on S. Use Green's theorem to solve $\nabla^2\varphi(\bar r) = -4\pi\rho(\bar r)$

subject to $\varphi(\bar r) = f(\bar r)$ $\forall \bar r$ on S.

Brief Review: We begin with the divergence theorem $\int_V \bar\nabla\cdot\bar A\, dV = \int_S \vec A\cdot\hat n\, dS$

and let $\bar A = \phi\nabla\psi$ $\quad \bar\nabla\cdot\bar A = \bar\nabla\phi\cdot\bar\nabla\psi + \phi\nabla^2\psi$ $\quad \vec A\cdot\hat n = \phi\bar\nabla\psi\cdot\hat n = \phi\frac{\partial\psi}{\partial n}$

so $\int_V \bar\nabla\phi\cdot\bar\nabla\psi + \phi\nabla^2\psi\, dV = \int \phi\frac{\partial\psi}{\partial n}\, dS$ interchanging ϕ and ψ and subtracting

$$\int_V \phi\nabla^2\psi - \psi\nabla^2\phi\, dV = \int\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS$$

choosing $\phi = \Phi$ and $\psi = \frac{1}{|\bar x - \bar x'|}$ and using $\nabla^2\Phi = -4\pi\rho$ $\quad \nabla^2\frac{1}{|\bar x - \bar x'|} = -4\pi\delta(x-x')$

yields $-4\pi\int_V \Phi(x)\delta(\bar x - \bar x') - \frac{\rho(x')}{|\bar x - \bar x'|}\, d^3x' = \int\left(\Phi\frac{\partial}{\partial n'}\frac{1}{|\bar x - \bar x'|} - \frac{1}{|\bar x - \bar x'|}\frac{\partial\Phi}{\partial n'}\right)da'$

Thus
$$\Phi(\bar x) = \int_V \frac{\rho(x')}{|\bar x - \bar x'|}\, d^3x' + \frac{1}{4\pi}\int_S\left(\frac{1}{|\bar x - \bar x'|}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\frac{1}{|\bar x - \bar x'|}\right)da'.$$

Now instead we choose $\phi = \Phi$ and $\psi = G(\bar x, \bar x')$ where $\nabla'^2 G(\bar x \bar x') = -4\pi\delta$

(this means $G(\bar x\bar x') = 1/|\bar x - \bar x'| + F(\bar x\bar x')$ where $\nabla'^2 F(\bar x\bar x') = 0$) we get

$$\Phi(\bar x) = \int_V \rho(\bar x')G(\bar x, \bar x')\, d^3x' + \frac{1}{4\pi}\int_S\left(G(\bar x\bar x')\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}G(\bar x\bar x')\right)da'$$

Now if $G(\bar x\bar x') = 0$ on S (This is called Dirichlet bdy conditions)

and if $\Phi(\bar x') = f(\bar x')$ on S, then

$$\boxed{\Phi(\bar x) = \int_V \rho(\bar x')G(\bar x\bar x')\, d^3x' + \frac{-1}{4\pi}\int f(\bar x')\frac{\partial}{\partial n'}G(\bar x, \bar x')\, da'}$$

(b) Consider the space $z > 0$. Bdy condition is $z = 0$ at $\Phi = 0$ except equilateral triangular region of side a $\Phi = V$. Find $G(\bar{x}, \bar{x}')$ and hence $\Phi(\bar{x})$. Evaluate first nonvanishing term as $|\bar{x}| \to \infty$.

Consider the function
$$\boxed{\begin{aligned} &G(\bar{x}, \bar{x}') \equiv \frac{1}{|\bar{x} - \bar{x}'|} - \frac{1}{|\bar{x} - \bar{x}''|} \\ &\text{where if } \bar{x}' = (x', y', z') \\ &\qquad \text{then } \bar{x}'' \equiv (x', y', -z') \end{aligned}}$$

This clearly has the property that when $z' = 0$ $G(\bar{x}, \bar{x}') = 0$ and also $\nabla^2 G(\bar{x}\, \bar{x}') = -4\pi\, \delta(\bar{x} - \bar{x}')$ in the region of interest $z' > 0$ (there is also a $\delta(\bar{x} - \bar{x}'')$ term $|\bar{x}' - \bar{x}''|$ never vanishes for $z' > 0$, $z > 0$).

Presumably $\rho(\bar{x}') = 0$. Also $\frac{\partial}{\partial n'} G(\bar{x}, \bar{x}') = \frac{\partial}{\partial(-z')} \left\{ \frac{1}{|\bar{x} - \bar{x}'|} - \frac{1}{|\bar{x} - \bar{x}''|} \right\}$

$$= -\frac{\partial}{\partial z'} \left\{ \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-1/2} - \left[(x-x')^2 + (y-y')^2 + (z+z')^2 \right]^{-1/2} \right\}$$

$$= -\left\{ -\tfrac{1}{2}\, 2(z-z')(-1) \Big/ |\bar{x} - \bar{x}'|^3 - \left(-\tfrac{1}{2}\right) 2(z+z') \Big/ |\bar{x} - \bar{x}''|^3 \right\}$$

where I have chosen $n = -z'$ because we must use outward pointing normal from volume $V \Leftrightarrow z > 0$ so this means the negative z direction.

On the surface $z' = 0$ we have $\bar{x}'' = \bar{x}'$ so $\frac{\partial G(\bar{x}, x')}{\partial n'} = \frac{-2z}{|\bar{x} - \bar{x}'|^3}$

Thus
$$\Phi(\bar{x}) = -\frac{1}{4\pi} \int_{\Delta} V \frac{-2z}{|\bar{x} - \bar{x}'|^3}\, dx'\, dy'$$

$$\boxed{\Phi(x, y, z) = \frac{Vz}{2\pi} \int_{\Delta} \frac{dx'\, dy'}{\left[(x-x')^2 + (y-y')^2 + z^2 \right]^{3/2}} \quad \text{where } \int_{\Delta} \text{ means an integration over the triangle}}$$

Scalettar

One question we can ask about this result is whether it reproduces properly the given potential on the plane $z=0$. At first glance it appears that we are in trouble because of the overall factor of z which might make $\Phi(x,y,z) \equiv 0$. This is not the case however because the denominator may also vanish. In fact when $x \neq x'$ or $y \neq y'$ we clearly get zero as $z \to 0$. But if $x=x'$ and $y=y'$ we get $z/z^3 = 1/z^2 \to \infty$ as $z \to 0$.

It is therefore clear that

$$z \Big/ \left((x-x')^2+(y-y')^2+z^2\right)^{3/2} \propto \delta(x-x')\delta(y-y')$$

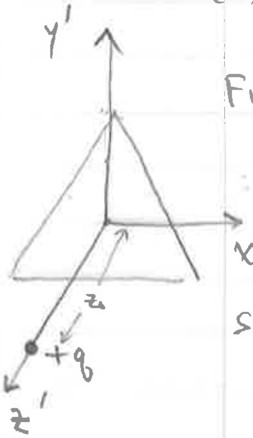
what we need, in fact, is for it to equal $2\pi\,\delta(x-x')\delta(y-y')$; then when we substitute $z=0$ we reproduce potential on plane.

To evaluate the first nonvanishing term as $|\vec{r}| \to \infty$ we simply ignore the x' and y' in the denominator of the integral on page 9 to get

$$\boxed{\Phi(x,y,z) \sim \frac{VzA}{2\pi r^3} \quad \text{where } A = a^2\sqrt{3}/24 = \text{area of triangle and } r = \sqrt{x^2+y^2+z^2} = \text{distance from triangle center}}$$

<u>Comment</u> Φ falls off very rapidly $\sim 1/r^3$! This is much faster than a point charge $\sim 1/r$. The point is that the huge infinite plane is held at $V=0$ which really makes Φ off the plane small also.

(c) A point charge q is added a distance z_0 above the triangle. Find an integral representation of Φ and evaluate as $|\vec{r}| \to \infty$. Interpret.



We no longer have $\rho(\vec{x}') = 0$ but rather $\rho(\vec{x}') = q\,\delta(x')\delta(y')\delta(z'-z_0)$

So now $\Phi(\vec{x}) = \int_V d^3x\, q\,\delta(x')\delta(y')\delta(z'-z_0)\left\{\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right\}$

$$+ \frac{Vz}{2\pi}\int_\Delta dx'dy' \, 1\Big/\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}$$

$$\boxed{\Phi(x,y,z) = \frac{q}{\sqrt{x^2+y^2+(z-z_0)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+z_0)^2}} + \frac{Vz}{2\pi}\int_\Delta dx'dy' \frac{1}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}}}$$

In limit $|\vec{r}| \to \infty$ we get

$$q\left[\left\{(x^2+y^2+z^2) - 2zz_0 + z_0^2\right\}^{-1/2} - \left\{(x^2+y^2+z^2) + 2zz_0 + z_0^2\right\}^{-1/2}\right] + \frac{VzA}{2\pi r^3}$$

$$\frac{q}{r}\left[\left(1 - \frac{2zz_0 \mp z_0^2}{r^2}\right)^{-1/2} - \left(1 + \frac{2zz_0 + z_0^2}{r^2}\right)^{-1/2}\right] + \frac{VzA}{2\pi r^3}$$

$$\frac{q}{r}\left[\cancel{1} + \frac{zz_0}{r^2} - \cancel{\frac{z_0^2}{2r^2}} + \dots - \cancel{1} + \frac{zz_0}{r^2} + \cancel{\frac{z_0^2}{2r^2}} + \dots\right] + \frac{VzA}{2\pi r^3}$$

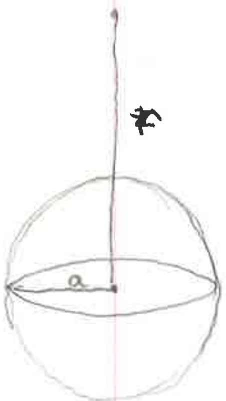
$$\Phi(x,y,z) \to \frac{2qz_0}{r^3} z + \frac{VzA}{2\pi r^3}$$

dipole term due to charge q and its image
Note p = dipole moment
$p = 2qz_0$
The potential due to dipole is $\frac{p\cos\theta}{r^2}$ but $z = r\cos\theta$
so this is correct form

Term due to potential V in triangular region

6. Nonconducting sphere radius a, total charge Q. $Q - Q_E$ uniform in volume Q_E in thin ring about equator. Q, Q_E same sign.

(a) Find $\varphi(\vec{r})$ on polar axis for $|\vec{r}| > a$



The charge $Q - Q_E$ distributed uniformly in sphere must by gauss give a potential $Q - Q_E / r$ [illegible]

The ring Q_E gives a potential $Q_E/(r^2+a^2)^{1/2}$

Thus $$\varphi(\vec{r}) = \frac{Q-Q_E}{r} + \frac{Q_E}{(r^2+a^2)^{1/2}}$$

where $r = |\vec{r}| =$ distance of point $\vec{r}$ from sphere center

(b) Express potential as power series in spherical coordinates and hence obtain $\varphi(\vec{r})$ $\forall\, \vec{r}$ with $|\vec{r}| > a$ even off polar axis!

$$\varphi(\vec{r}) = \frac{Q-Q_E}{r} + \frac{Q_E}{r}\left(1 + a^2/r^2\right)^{-1/2}$$

$$= \frac{Q-Q_E}{r} + \frac{Q_E}{r}\left(1 - \frac{1}{2}\frac{a^2}{r^2} + \frac{3}{8}\frac{a^4}{r^4} - \dots\right)$$

$$= \frac{Q}{r} + Q_E\left(-\frac{1}{2}\frac{a^2}{r^3} + \frac{3}{8}\frac{a^4}{r^5} - \dots\right)$$

Now according to the prescription in Jackson, we append a factor of $P_\ell(\cos\theta)$ to any term in $1/r^{\ell+1}$,

$$\boxed{\varphi(\vec{r}) = \frac{Q}{r} + Q_E\left(-\frac{1}{2}\frac{a^2}{r^3}P_2(\cos\theta) + \frac{3}{8}\frac{a^4}{r^5}P_4(\cos\theta) + \dots\right)}$$

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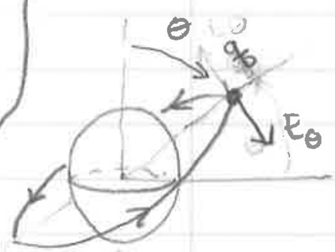
(c) calculate $\vec{E}$ for large $\vec{r}$ to order a^2/r^2. Consider particle of charge q (of opposite sign to Q) held in bound orbit. Show orbit is stable in inclination, the angle between orbit plane and equatorial plane) only for zero inclination. Discuss grav. analog.

$$\vec{E} = -\nabla\varphi(\vec{r}) = -\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{Q}{r} - \frac{Q_E a^2}{2}\frac{1}{r^3}\frac{1}{2}(3\cos^2\theta - 1) + \dots\right)$$

$$\vec{E} = -\hat{r}\left(-\frac{Q}{r^2} + \frac{3Q_E a^2}{2 r^4}(3\cos^2\theta - 1) + \dots\right) + \hat{\theta}\left(+\frac{Q_E a^2}{4r^4}\, 3\, 2\cos\theta\sin\theta + \dots\right)$$

$$\boxed{\vec{E} = \frac{Q}{r^2}\left[1 - \frac{3Q_E}{4Q}\frac{a^2}{r^2}(3\cos^2\theta - 1) + \dots\right]\hat{r} + \frac{Q_E}{r^2}\left[\frac{3\sin 2\theta}{4}\frac{a^2}{r^2} + \dots\right]\hat{\theta}}$$

Now consider the electric field in the $\hat{\theta}$ direction



For the orbit to be stable in inclination, the condition is clearly first that $E_\theta = 0$ so that there be no force moving q out of orbital plane. This condition is satisfied only by $\theta = 0$ and $\theta = \pi/2$. ✓

But for stability we furthermore need that in a small displacement from $\theta = \theta_0$ the force E_θ restores back to θ_0.

use:

For $\theta = 0$ $\quad E_\theta(d\theta) \cong Q_E/r^2\left(\frac{3\,2d\theta}{4}\, a^2/r^2\right)$

$\sin(2d\theta) \sim 2d\theta$

This is not stable since E_θ tends to push us further away from $\theta_0 = 0$

For $\theta = \pi/2$ $\quad E_\theta(d\theta) = Q_E/r^2\left(\frac{3(-2d\theta)}{4}\, a^2/r^2\right)$

$\sin 2(\pi/2 + d\theta)$

$\cong \sin(\pi + 2d\theta)$

$\sim -2d\theta$

This is stable since E_θ tends to push us back towards $\theta_0 = \pi/2$ i.e. E_θ has opposite sign from $d\theta$.

The gravitational analog is clear. This situation models the equatorial bulge of the earth and we see that the only stable orbit for a satellite is thus around the equator.