$$H_0 = -t\sum(c_\ell^\dagger c_{\ell+1} + c_{\ell+1}^\dagger c_\ell)$$

In space of $N=1$ electron basis is

$|1000\ldots\rangle$

$|0100\ldots\rangle$

Matrix of H is
$$\begin{pmatrix} 0 & -t & & \\ -t & 0 & -t & \\ & -t & 0 & \\ & & & \ddots \end{pmatrix}$$

Eigenvalues $-2t\cos k$

What is $V = \epsilon_A \sum_{\ell\,odd} c_\ell^\dagger c_\ell + \epsilon_B \sum_{\ell\,even} c_\ell^\dagger c_\ell$ added

$$\begin{pmatrix} \epsilon_A & -t & & \\ -t & \epsilon_B & -t & \\ & -t & \epsilon_A & \\ & & & \ddots \end{pmatrix}$$

<u>OLD</u> Method from phonon problem

$$(H_0 + V)\psi = \lambda\psi$$

$$\psi_\ell = \begin{cases} Ae^{ik\ell} & \ell \text{ odd} \\ Be^{ik\ell} & \ell \text{ even} \end{cases}$$

KS-2

ℓ odd $\quad \epsilon_A A e^{ik\ell} - tBe^{ik(\ell+1)} - tBe^{+ik(\ell-1)} = \lambda A e^{ik\ell}$

ℓ even $\quad \epsilon_B B e^{ik\ell} - tAe^{ik(\ell+1)} - tAe^{ik(\ell-1)} = \lambda B e^{ik\ell}$

$$\epsilon_A A - 2t\cos k\, B = \lambda A$$

$$\epsilon_B B - 2t\cos k\, A = \lambda B$$

$$\begin{vmatrix} \epsilon_A - \lambda & -2t\cos k \\ -2t\cos k & \epsilon_B - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - (\epsilon_A + \epsilon_B) + \epsilon_A\epsilon_B - 4t^2\cos^2 k = 0$$

$$\lambda = \frac{1}{2}\left[(\epsilon_A + \epsilon_B) \pm \sqrt{\underbrace{(\epsilon_A + \epsilon_B)^2 - 4(\epsilon_A\epsilon_B - 4t^2\cos^2 k)}_{(\epsilon_A - \epsilon_B)^2 + 16t^2\cos^2 k}}\right]$$

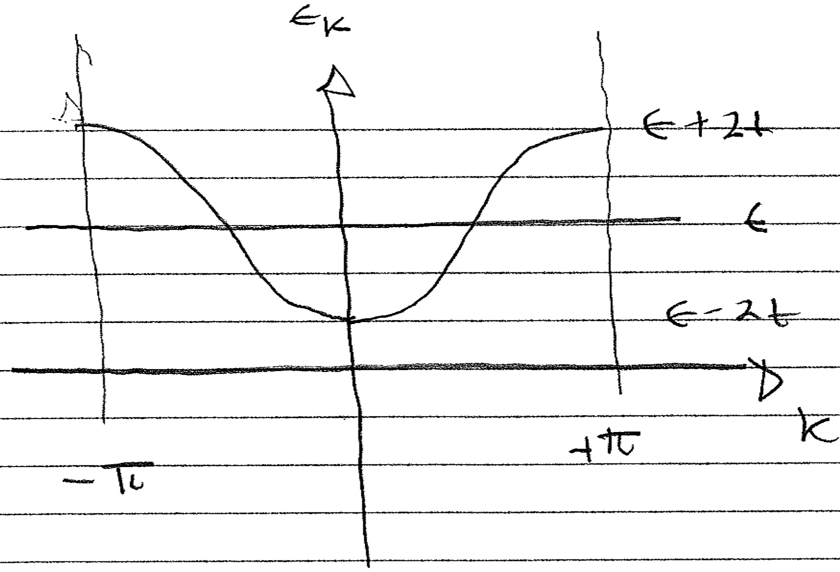
or $\quad \epsilon_A = \epsilon + \Delta \qquad \epsilon_A + \epsilon_B = 2\epsilon$

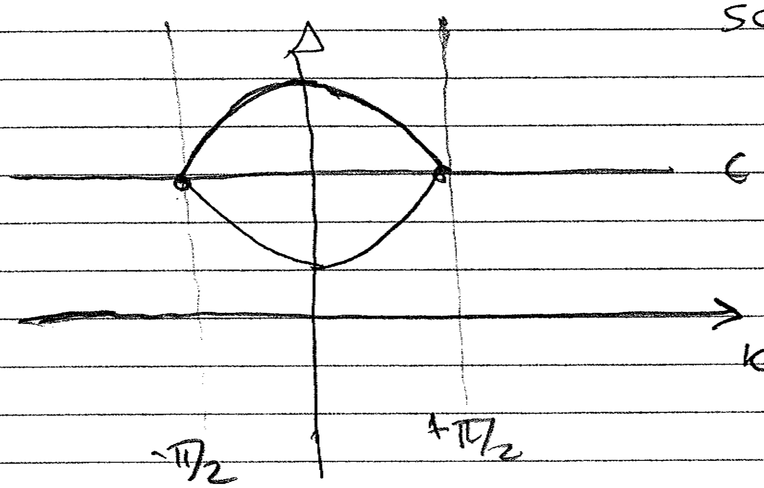
$\quad \epsilon_B = \epsilon - \Delta \qquad \epsilon_A - \epsilon_B = 2\Delta$

$$\lambda = \epsilon \pm \sqrt{\Delta^2 + 4t^2\cos^2 k}$$

$\Delta = 0$ problem $\quad \lambda = \epsilon - 2t\cos k$

KS - 3



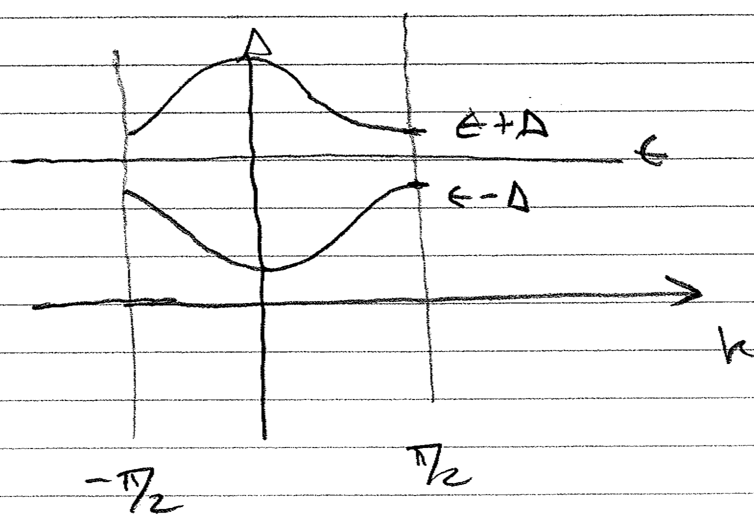


same list of eigenvalues

Just assign 2 ϵ_k

to each of ½

the k points



$\epsilon \pm \sqrt{\Delta^2 + 4t^2 \cos^2 k}$

This approach is nice, but it conceals the impt physics which is that

$$V = \epsilon \sum c_\ell^\dagger c_\ell + \Delta \sum_\ell (-1)^\ell c_\ell^\dagger c_\ell$$

induces scattering between e^- of momentum k and $k+\pi$.

This scattering can be viewed as inhibiting motion of e^- which has momentum $\pi/2$ since returns it to $-\pi/2$ which precisely cancels out the motion

Let's do the problem in a way which better exhibits this physics!

c_ℓ, c_ℓ^+ destroy/create e^- at <u>spatial</u> <u>site</u> ℓ

What's special about creating/destroying at a particular location in space? Why not create/destroy e^- of particular <u>momentum</u>?

$$c_k^+ = \frac{1}{\sqrt{N}} \sum_\ell e^{ik\ell} c_\ell^+ \qquad a =$$

$$k = \frac{2\pi}{N}\{1, 2, \ldots N\}$$

$$c_k = \frac{1}{\sqrt{N}} \sum_\ell e^{-ik\ell} c_\ell \qquad \text{or} \quad \frac{2\pi}{N}\left\{-\frac{N}{2}+1, \ldots \frac{N}{2}\right\}$$

Fourier transform $f(x) = f(x+L)$

$$f(x) = \sum_{n=0}^{\infty} a_n \sin\frac{2\pi n}{L}x + b_n \cos\frac{2\pi n}{L}x$$

$$a_n = \frac{2}{L}\int_0^L f(x) \sin\frac{2\pi n}{L}x \, dx$$

Because $\int_0^L \sin\frac{2\pi n x}{L} \sin\frac{2\pi m x}{L} dx = \frac{L}{2}\delta_{\ell m}$

$$\int_0^L \sin\frac{2\pi n x}{L} \cos\frac{2\pi m x}{L} dx = 0$$

sin, cos are orthogonal

$$\int_0^L \sin\frac{2\pi m x}{L}\, f(x)\,dx = \sum_n a_n \frac{L}{2}\delta_{\ell m} = \frac{L}{2} a_m$$

Could also use

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi n/_L x}$$

$$\int_0^L e^{i2\pi n/_L x} e^{-i2\pi m x/_L}\,dx = \delta_{\ell m} L$$

$$c_n = \frac{1}{L}\int_0^L e^{-i2\pi n x/_L} f(x)\,dx$$

By analogy

$$c_\ell^+ = \frac{1}{\sqrt{N}}\sum_k e^{-ik\ell} c_k^+$$

guess inversion formula

$$c_\ell = \frac{1}{\sqrt{N}}\sum_k e^{ik\ell} c_k$$

Proof

$$c_k^+ = \frac{1}{\sqrt{N}}\sum_\ell e^{ik\ell} c_\ell^+$$

$$\sum_k e^{-ikj} c_k^+ = \frac{1}{\sqrt{N}}\sum_\ell \underbrace{\sum_k e^{ik(\ell-j)}}_{N\delta_{\ell j}} c_\ell^+$$

Discussed this before...

$$= \sqrt{N}\, c_j^+$$

$$\therefore\ c_j^+ = \frac{1}{\sqrt{N}}\sum_k e^{-ikj} c_k^+ .$$

What might you worry about?

$$\{c_\ell^\dagger, c_{\ell'}^\dagger\} = \delta_{\ell\ell'} \iff \text{PAULI + ANTISYM}$$

$$\{c_k^\dagger, c_{k'}^\dagger\} = \frac{1}{N}\sum_\ell \sum_{\ell'} e^{ik\ell} e^{ik\ell'} \underbrace{\{c_\ell^\dagger, c_{\ell'}^\dagger\}}_{= \phi} = \phi$$

Trickier: $\{c_\ell^\dagger, c_{\ell'}\} = \delta_{\ell\ell'}$

$$\{c_k^\dagger, c_{k'}\} = \frac{1}{N}\sum_\ell \sum_{\ell'} e^{ik\ell} e^{-ik'\ell'} \underbrace{\{c_\ell^\dagger, c_{\ell'}\}}_{\delta_{\ell\ell'}}$$

$$= \frac{1}{N}\sum_\ell e^{i(k-k')\ell} = \delta_{k,k'} \;!$$

$$H = -t \sum_{\ell} \left(c_\ell^{\dagger} c_{\ell+1} + c_{\ell+1}^{\dagger} c_\ell \right)$$

Guesses ??

$$-t \sum_{\ell} \sum_{k} \sum_{k'} \left(e^{ik\ell} e^{-ik'(\ell+1)} + e^{ik(\ell+1)} e^{-ik'\ell} \right) c_k^{\dagger} c_{k'}$$

$$e^{i(k-k')\ell} \left[e^{-ik'} + e^{ik} \right] c_k^{\dagger} c_{k'}$$

$\sum_{\ell}$ gives $\delta_{kk'}$

$$-t \sum_{k} 2\cos k \, c_k^{\dagger} c_k$$

$$H = \sum -2t\cos k \, c_k^{\dagger} c_k$$

Work in $1 e^-$ sector

$\# e^-$ of momentum k_1 etc

$$|100\ldots\rangle$$
$$|010\ldots\rangle$$
$$|001\ldots\rangle$$

$$H = \begin{pmatrix} -2t\cos k_1 & & \\ & -2t\cos k_2 & \\ & & \ddots \end{pmatrix}$$

KS-9

Now consider $V = \epsilon \sum_l c_l^\dagger c_l + \Delta \sum_l (-1)^l c_l^\dagger c_l$

$$\frac{1}{N}\epsilon \sum_l \sum_k \sum_{k'} \underbrace{e^{ikl} e^{-ik'l}}_{\delta_{kk'}} c_k^\dagger c_{k'}$$

$$= \epsilon \sum_k c_k^\dagger c_k$$

$\epsilon - 2t\cos k$ for $\Delta = 0$

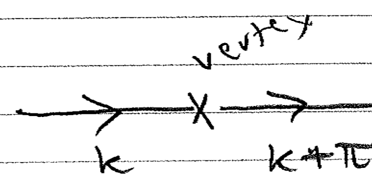
$$\Delta \sum_l (-1)^l c_l^\dagger c_l$$

$$\frac{1}{N} \Delta \sum_l (-1)^l \sum_k \sum_{k'} c_k^\dagger c_{k'} e^{ikl} e^{-ik'l}$$ now what?!

$(-1)^l = e^{i\pi l}$

$$\frac{1}{N} \Delta \sum_k \sum_{k'} \underbrace{\sum_l e^{i\pi l} e^{ikl} e^{-ik'l}}_{N\delta_{k+\pi,k'}} c_k^\dagger c_{k'}$$

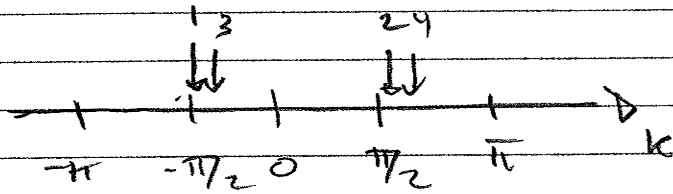
$$= \Delta \sum_k c_k^\dagger c_{k+\pi}$$



"Feynman diagram"

k and k+π mix with each other but with nothing else

Arrange matrix so k, $k+\pi$ adjacent



$$\begin{pmatrix} \epsilon - 2t\cos k_1 & \Delta \\ \Delta & \epsilon - 2t\cos(k_1+\pi) \end{pmatrix}$$

$$-2t\cos(k_1+\pi) = 2t\cos k_1$$

Eigenvalues $\begin{vmatrix} \epsilon - 2t\cos k_1 - \lambda & \Delta \\ \Delta & \epsilon + 2t\cos k_1 - \lambda \end{vmatrix} = 0$

$\lambda = \epsilon \pm \sqrt{\Delta^2 + 4t^2\cos^2 k}$ as in previous method!