FT-1

Fermions at finite T

Review Bosons

Energy level $\hbar\omega$
occupied by n quanta
$n = 0, 1, 2, .. \infty$

$$E_n = (n + 1/2)\hbar\omega$$

$$Z = \sum_n e^{-\beta E_n} = e^{-\beta 1/2 \hbar\omega} \sum_n (e^{-\beta\hbar\omega})^n$$

$$= e^{-\beta \frac{1}{2}\hbar\omega}(1 - e^{-\beta\hbar\omega})^{-1}$$

$$\langle E \rangle = -\frac{\partial}{\partial\beta} \ln Z = -\frac{\partial}{\partial\beta}\left[-\beta\hbar\omega/2 - \ln(1 - e^{-\beta\hbar\omega})\right]$$

$$= \frac{1}{2}\hbar\omega + \frac{1}{1 - e^{-\beta\hbar\omega}} \hbar\omega e^{-\beta\hbar\omega}$$

$$= \hbar\omega\left[\underbrace{\frac{1}{e^{\beta\hbar\omega} - 1}}_{\langle n \rangle} + \frac{1}{2}\right]$$

$$\langle n \rangle_{BE} = \frac{1}{e^{\beta\hbar\omega} - 1}$$

Chemical potential μ $\qquad E_n = (n + 1/2)\hbar\omega - \mu n$

$\mu = \partial\langle E \rangle / \partial N$ energy cost to add particle to system

$$\langle n \rangle_{BE} = \frac{1}{e^{\beta(\hbar\omega - \mu)} - 1}$$

project:
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Fermions are simpler. If there is an energy level ϵ it can only have $n=0$ or $n=1$

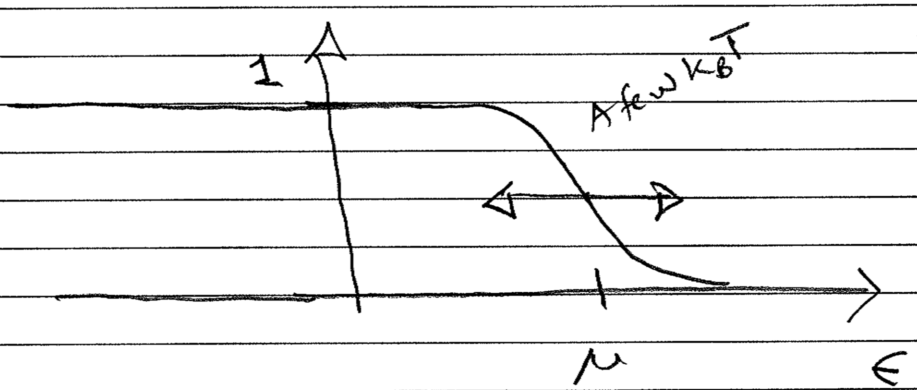
$$Z = 1 + e^{-\beta\epsilon} \qquad E = \epsilon n$$

$$\langle E \rangle = -\frac{\partial}{\partial\beta} \ln Z = \frac{1}{1+e^{-\beta\epsilon}} \epsilon e^{-\beta\epsilon} = \frac{\epsilon}{e^{\beta\epsilon}+1}$$

$$\langle n \rangle_{FD} = \frac{1}{e^{\beta\epsilon}+1}$$

Again introduce chemical potential $\quad E = \epsilon n - \mu n$

$$\langle n \rangle_{FD} = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$



$\epsilon - \mu < 0$

eg $\beta(\epsilon-\mu) = -4$

$\langle n \rangle_{FD} \approx 1$

$\epsilon - \mu > 0$

eg $\beta(\epsilon-\mu) = +4$

$\langle n \rangle_{FD} \approx 0$

At $T=0$ $\mu = E_F$

The highest energy level occupied

$\epsilon - \mu = -4/\beta$

$\epsilon = \mu - 4k_BT \quad n \sim 1$

$\epsilon = \mu + 4k_BT \quad n \sim 0$

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Recall $E_F \sim 3eV \sim 5 \cdot 10^{-19}$ J

$k_B T \sim (1.38 \cdot 10^{-23})(300) \sim 4 \cdot 10^{-21}$ J

So $k_B T \ll E_F$

"Sommerfeld Expansion" Ashcroft Mermin Appendix C

Expansion in ratio $k_B T / E_F$

$$\int_{-\infty}^{\infty} H(\epsilon) \frac{1}{e^{\beta(\epsilon-\mu)}+1} d\epsilon$$

$$= \int_{-\infty}^{\mu} H(\epsilon) d\epsilon + \sum_{n=1}^{\infty} a_n (k_B T)^{2n} \frac{d^{2n-1}}{d\epsilon^{2n-1}} H(\epsilon) \Big|_{\epsilon=\mu}$$

$$a_n = 2\left(1 - 1/2^{2n} + 1/3^{2n} - 1/4^{2n} + 1/5^{2n} \ldots\right)$$

(can be related to Riemann zeta function)

Bernoulli #'s etc

This can be used to show precise value for electronic contribution to specific heat is:

$C = \frac{1}{2}\pi^2 N k_B T / T_F$ $\quad k_B T_F = E_F$

DIVOGA

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$n = N/V$ density

Density of states of free electrons

$$E = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3} \qquad \text{(page 7)}$$

highest Energy filled

$$N = \frac{V}{3\pi^2}\left(\frac{2mE}{\hbar^2}\right)^{3/2}$$

Density of states $D(E) = \frac{dN}{dE} = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}$