

1. (a) Show that the kinetic energy of a three-dimensional gas of  $N$  free electrons at 0 K is:  $U_0 = \frac{3}{5} N \epsilon_F$ . (b) Derive a relation connecting the pressure and volume of an electron gas at 0 K. (c) Show that the bulk modulus  $B = -V \frac{\partial P}{\partial V}$  of an electron gas at 0 K is  $B = 5P/3 = 10U_0/9V$ . (d) Estimate for potassium, using Table 1, the value of the electron gas contribution to  $B$ .

2. Use the equation  $m \left( \frac{dv}{dt} + v/\tau \right) = -eE$  for the electron drift velocity  $v$  to show that the conductivity at frequency  $\omega$  is:

$$(1) \quad \sigma(\omega) = \sigma(0) \left[ \frac{1 + i\omega\tau}{1 + (\omega\tau)^2} \right].$$

3. For the drift velocity theory of Eq. 51 in the text, show that the static current density can be written in matrix form as:

$$(2) \quad \begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \frac{1 + (\omega_c\tau)^2}{\sigma_0} \begin{pmatrix} 1 & \omega_c\tau & 0 \\ -\omega_c\tau & 1 & 0 \\ 0 & 0 & 1 + \omega_c\tau^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

- In the high magnetic field limit of  $\omega_c\tau \gg 1$ , show that  $\sigma_{yx} = \frac{B}{ne c} = \sigma_{xy}$ . In this limit,  $\sigma_{xx} = 0$ , to order  $1/\omega_c\tau$ . The quantity  $\sigma_{xy}$  is called the **Hall conductivity**.

① KE of free electron gas in 3D is:

$$(a) U_0 = \sum_k \epsilon_k f(\epsilon_k) = \int_0^{\epsilon_F} \epsilon D(\epsilon) d\epsilon$$

but in 3D,  $D(\epsilon) = \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \epsilon^{1/2}$ , so

$$U_0 = \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\epsilon_F} \epsilon^{3/2} d\epsilon$$

$$= \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \left[ \frac{2}{5} \epsilon^{5/2} \right]_0^{\epsilon_F} = \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{2}{5} \epsilon_F^{5/2}$$

but  $N = \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \left[ \frac{2}{3} \epsilon^{3/2} \right]_0^{\epsilon_F}$ , so re-writing we have

$$U_0 = \frac{3}{5} N \frac{2}{3} \epsilon_F = \frac{3}{5} N \epsilon_F$$

(b) pressure  $\equiv - \frac{\partial U}{\partial V}$  now find  $\epsilon_F$  in terms of  $V$ :

$$\epsilon_F = \frac{2\pi^2}{V} \left( \frac{2m}{\hbar^2} \right)^{3/2} N$$

$$U_0 = \frac{1}{5} \frac{2\pi^2}{V} \left( \frac{2m}{\hbar^2} \right)^{3/2} N \left( \frac{V}{2\pi^2 N} \right)^{5/3} \left( \frac{2\pi^2 N}{V} \right)^{5/3}$$

$$= \frac{1}{5} \frac{2\pi^2}{V} \left( \frac{2m}{\hbar^2} \right)^{3/2} N \left( \frac{2\pi^2 N}{V} \right)^{5/3} \left( \frac{V}{2\pi^2 N} \right)^{5/3}$$

$$1 - \frac{2}{3} = \frac{3}{3} - \frac{2}{3} = \frac{1}{3}$$

$$= \frac{3}{2} \left( 1.4 \times 10^{22} \text{ cm}^{-3} \right) (2.12 \text{ eV})$$

$$= 2.0 \times 10^{22} \text{ eV/cm}^3$$

Therefore,  $B = \frac{9}{10} \frac{U_0}{V} = \frac{9}{10} \cdot \frac{5}{3} \frac{V}{N} \epsilon_F$

(d) for K, we have:  $\frac{V}{N} = 1.4 \times 10^{22} \text{ cm}^{-3}$  and  $\epsilon_F = 2.12 \text{ eV}$

So  $B = -V \frac{\partial P}{\partial V} = \boxed{\frac{3}{5} P} = \frac{3}{5} \cdot \frac{2}{3} \frac{U_0}{V} = \boxed{\frac{2}{5} \frac{U_0}{V}}$

$$= -\frac{5}{3} P V^{-1}$$

$$\frac{\partial P}{\partial V} = -\frac{3}{2} \left( \frac{1}{5\pi^2} \right) \left( \frac{h^2}{2m} \right) (3\pi^2 N)^{2/3} \left( -\frac{3}{5} \right) V^{-5/3}$$

(e) Bulk modulus:  $B = -V \left( \frac{\partial P}{\partial V} \right)$

So  $\boxed{P = \frac{3}{2} \frac{U_0}{V}}$

$$= +\frac{3}{2} U_0 V^{-1}$$

So  $P = -\left( -\frac{3}{2} \right) \frac{1}{5\pi^2} \left( \frac{h^2}{2m} \right) (3\pi^2 N)^{2/3} V^{-5/3}$

but  $1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$   
 $1 \text{ cm}^{-3} = 1 \times 10^{-6} \text{ m}^{-3}$

ad  $1 \text{ Pa} = 1 \text{ N/m}^2 = 1 \text{ J/m}^3$

so  $B = 2.0 \times 10^{22} \frac{\text{eV}}{\text{cm}^3} \cdot \frac{1.6 \times 10^{-19} \text{ J/eV}}{10^{-6} \text{ m}^3/\text{cm}^3} = 3.17 \times 10^9 \text{ Pa}$

or  $B = 3.17 \text{ GPa}$

②

$m \left( \frac{dv}{dt} + \frac{v}{\tau} \right) = -eE$

$j = -nev$  current density

but if  $\vec{E} = \vec{E}_0 e^{-i\omega t}$

$v = v_0 e^{-i\omega t}$

we have  $m(-i\omega v_0 + \frac{1}{\tau} v_0 e^{-i\omega t}) = -eE_0 e^{-i\omega t}$

or  $-eE = \frac{m}{\tau} v (1 - i\omega\tau)$

$v = \frac{m(1 - i\omega\tau)}{-e\tau} E$

$E \frac{m(1 + i\omega\tau)}{2m\tau + 1} \frac{m}{-e\tau} =$

$$\begin{pmatrix} E_z \\ E_y \\ E_x \end{pmatrix} \frac{m}{2} = \overbrace{\begin{pmatrix} 1-i\omega\tau & 0 & 0 \\ 0 & 1-i\omega\tau & 0 \\ 0 & 0 & 1-i\omega\tau \end{pmatrix}}^W \begin{pmatrix} V_z \\ V_y \\ V_x \end{pmatrix}$$

$$\left\{ \begin{aligned} 0 V_x + 0 V_y + \frac{m}{2} V_z &= E_z \\ -\frac{c}{B} V_x + \frac{m}{2} (1-i\omega\tau) V_y + 0 V_z &= -E_y \\ \frac{c}{B} V_x + \frac{m}{2} (1-i\omega\tau) V_y + 0 V_z &= -E_x \end{aligned} \right.$$

or if we write  $V = V e^{i\omega t}$ , then

$$m \left( \frac{d}{dt} + 1 \right) \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} -c \left( E_x + \frac{c}{B} V_y \right) \\ -c \left( E_y - \frac{c}{B} V_x \right) \\ -E_z \end{pmatrix}$$

③ from Eq. 51 we have:

$$\sigma(\omega) = \frac{ne^2 \tau}{1 + i\omega\tau} \underbrace{\frac{m}{1 + i\omega\tau}}_{\sigma(\omega)}$$

therefore

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{22^2 m + 1}{1} & \frac{22^2 m + 1}{2^2 m + 1} \\ 0 & \frac{22^2 m + 1}{2^2 m - 1} & \frac{22^2 m + 1}{1} \end{pmatrix} = M^{-1} \frac{1}{2}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2^2 m - 1 \\ 0 & 2^2 m & 1 \end{pmatrix} = M$$

but note that we only care about the static component (i.e.,  $\omega = 0$ ), so

$$\vec{J} = -en\vec{v} = \frac{ne^2}{m} M^{-1} \vec{E}$$

and hence

$$\vec{v} = M^{-1} \left( \frac{ne^2}{2\tau} \right) \vec{E}$$

Solving for  $\vec{v}$  we have

$$\boxed{\frac{B}{-nec} =}$$

$$\sigma_{xy} = -\sigma_{yx} = -\frac{e^2 n \hbar}{mc} \cdot \frac{B}{1}$$

$$\text{for } \omega_c \tau \gg 1, \quad \sigma_{xy} = \frac{1 + \omega_c^2 \tau^2}{\omega_c \tau} \approx \frac{\omega_c \tau}{1}$$

$$\sigma = \frac{1 + \omega_c^2 \tau^2}{\omega_c \tau} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\omega_c \tau}{1 + \omega_c^2 \tau^2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$