Identical Particles

In classical mechanics, we can treat particles completely independently if they do not interact.

\[ V(\vec{r}_1, \vec{r}_2) = V(\vec{r}_1) + V(\vec{r}_2) \]

\[ \vec{F}_1 = -\nabla V(\vec{r}_1, \vec{r}_2) = -\nabla_1 V \]

\[ \vec{F}_2 = -\nabla_2 V(\vec{r}_1, \vec{r}_2) = -\nabla_2 V \]

\[ m\vec{v}_1 = \vec{F}_1 \quad \text{same as if } \vec{F}_1 \text{ did not exist!} \]

\[ m\vec{v}_2 = \vec{F}_2 \]

It is not quite so simple in Q.M. Even without interactions, quantum particles affect each other. How?

[Pauli]

We can understand periodic table from H atom (no interactions) but if we assume Pauli, the periodic table requires this. Otherwise, a carbon atom could have all its electrons in \( R_1(r)Y_0(\theta, \phi) \).
\[ -\frac{\hbar^2}{2m_1} \nabla_1^2 \psi - \frac{\hbar^2}{2m_2} \nabla_2^2 \psi + V(r_1, r_2) \psi = E \psi \]

\[ V_a(r_1) + V_b(r_2) \]

Try \( \psi(r_1, r_2) = \psi_a(r_1) \psi_b(r_2) \)

Solve individually:
\[
\left[ -\frac{\hbar^2}{2m_1} \nabla_1^2 + V_a \right] \psi_a(r_1) = E_a \psi_a(r_1)
\]

\[
\left[ -\frac{\hbar^2}{2m_2} \nabla_2^2 + V_b \right] \psi_b(r_2) = E_b \psi_b(r_2)
\]

And try \( \psi(r_1, r_2) = \psi_a(r_1) \psi_b(r_2) \)

Does it work? \[\boxed{\text{YES}}\]

But if particles are identical, doesn't really make sense.

\[ \psi_+ (r_1, r_2) \propto \psi_a(r_1) \psi_b(r_2) + \psi_a(r_2) \psi_b(r_1) \]

If \( a = b \) (bosons) \[ \psi_+ (r_1, r_2) \propto \psi_a(r_1) \psi_a(r_2) \]

\[ \psi_- (r_1, r_2) = 0 \rightarrow \text{PAULI} \]

(fermion)
Notice, as a consequence

\[ \Psi(r_1, r_2) = -\Psi(r_2, r_1) \] \text{ fermions, antisymmetric}

\[ \Psi(r_1, r_2) = \Psi(r_2, r_1) \] \text{ bosons, symmetric}

Actually, this is fundamental principle leading to choice

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Identical particles in 1d a square well

\[ \Psi_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \]

\[ E_n = \frac{\hbar^2 (n\pi)^2}{2ma^2} = \frac{n^2\hbar^2}{2ma^2} \geq kn^2 \]

Distinguishable particles (eg, an electron and a proton)

ground state \[ \Psi(x_1, x_2) = \frac{1}{a} \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{a} \quad E_{11} = 2k \]

excited states \[ \Psi(x_1, x_2) = \frac{1}{a} \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{a} \quad E_{12} = 5k \]

\[ \frac{1}{a} \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{a} \quad E_{21} = 5k \]
Bosons: same ground state. (Note \(4a(v_1)4b(v_2) + 4a(v_2)4b(v_1)\) = 2 \(4a(v_1)4b(v_2)\))

if \(a = b\)

\[E = 2k\]

But only one first excited state

\[\psi(x_1, x_2) = \frac{\sqrt{2}}{a} \left( \sin \frac{\pi x_1}{a} \sin \frac{2\pi x_2}{a} + \sin \frac{2\pi x_1}{a} \sin \frac{\pi x_2}{a} \right)\]

\[\frac{\pi}{a} \left( \frac{\pi}{a} + \frac{\pi}{a} \right) \nabla^2 \text{ why no crossing?}\]

\[E = 5k\]

Fermions ground state is \(E = 5k\)

\[\psi(x_1, x_2) = \frac{\sqrt{2}}{a} \left( \sin \frac{\pi x_1}{a} \sin \frac{2\pi x_2}{a} - \sin \frac{2\pi x_1}{a} \sin \frac{\pi x_2}{a} \right)\]

Idea behind

This is basis of high IEE of \(\varphi^+\) in metal!

ground state energy \(5k\) fermion \(2k\) bosons
Energy levels

Highest occupied level called Fermi energy $E_F$

Fermions: At most 2 per level

Bosons can all be put here (BEC at low $T$)

What would wave function be

$$\psi(x_1, x_2) = \psi_a(x_1) \psi_b(x_2) - \psi_a(x_2) \psi_b(x_1)$$

$$= \begin{vmatrix} \psi_a(x_1) & \psi_b(x_1) \\ \psi_a(x_2) & \psi_b(x_2) \end{vmatrix} \quad \text{"Slater determinant"}$$

$$\psi(x_1, x_2, x_3) = \begin{vmatrix} \psi_a(x_1) & \psi_b(x_1) & \psi_c(x_1) \\ \psi_a(x_2) & \psi_b(x_2) & \psi_c(x_2) \\ \psi_a(x_3) & \psi_b(x_3) & \psi_c(x_3) \end{vmatrix}$$

$$= \psi_a(x_1) \psi_b(x_2) \psi_c(x_3) - \psi_a(x_1) \psi_b(x_3) \psi_c(x_2) + \text{111} \quad \text{Minus sign upon } x_2 \rightarrow x_3$$

Why mathematically?
$e^-$ in metal can be treated as non-interacting because they are at such high density!!

Typical density of $e^-$ in metal $\approx 10^{22}/\text{cm}^3$

Energy levels $\frac{\hbar^2 k^2}{2m}$ but can put at most 2 $e^-$ in each (spin) $\rightarrow$

$E_0 \neq 0$ cannot put all in $k=0$

$N = \frac{2V}{(2\pi)^3} \int \text{d}^3k$

\[ \frac{\sqrt{\text{spin}}}{\text{seen my?}} \]

One motivation is dimensional

\[ \phi = \frac{N}{V} = \frac{k_F^3}{2\pi^2} \]

\[ k_F = \left( \frac{3\pi^2 N}{2} \right)^{\frac{1}{3}} \]

\[ kE \leftrightarrow E_F = \frac{\hbar^2 k^2}{2m} \sim k^2 \left( \frac{3\pi^2 N}{2} \right)^{2/3} \sim N^{2/3} \]

Can show $\langle KE \rangle = \frac{3}{5} E_F$
\[ \psi_k = e^{i \mathbf{k} \cdot \mathbf{r}} + \text{PBC} \]

\[ \mathbf{k} = \frac{2\pi n_x}{L} \quad \mathbf{k}_y = \frac{2\pi n_y}{L} \quad \mathbf{k}_z = \frac{2\pi n_z}{L} \]

\[ n_x, n_y, n_z \text{ integers} \]

Volume per k point is \[
\left( \frac{2\pi}{L} \right)^3
\]

ZP picture: \# points in k space:

\[
15 = \frac{1}{(2\pi L)^3} \int d^3k
\]

\[
= \frac{L^3}{(2\pi)^3} \left( \frac{L}{2\pi} \right)^3 \int d^3k = \frac{V}{(2\pi)^2} \int d^3k.
\]
What about PE?

PE ~