Gauge Transformations in QM

Given a wave function $\Psi(\vec{r})$ we can multiply by any phase and not change physics

$\Psi(\vec{r}) \rightarrow e^{i\theta} \Psi(\vec{r})$

$\left< 0 \right| = \int d^3 r \, \Psi(\vec{r}) e^{-i\theta \hat{O}} e^{i\theta \Psi(\vec{r})} = \left< 0 \right|_{\theta=0}$

Any observable pass through any observable

"local gauge transformation" "global gauge transformation"

Suppose $\theta = \Theta(\vec{r})$ instead of being constant then we cannot pass $e^{i\Theta(\vec{r})}$ through $\hat{O}$ eg if it has any momentum operators (derivatives).

$\hat{p}(e^{i\Theta(\vec{r})}) = \frac{\hbar}{i} \nabla (e^{i\Theta(\vec{r})} \Psi)$

$= \hbar \nabla \Theta \Psi + \frac{\hbar}{i} e^{i\Theta(\vec{r})} \nabla \Psi$

$\hat{\Theta}$

extra term
We would get \( \langle 0 \rangle_{\theta \neq 0} = \langle 0 \rangle_{\theta = 0} \).

If we insist that whenever we do local gauge transformation \( \Psi(\mathbf{r}) \rightarrow e^{i\phi(\mathbf{r})} \Psi(\mathbf{r}) \),

at the same time we change

\[
\mathbf{p} \rightarrow \mathbf{p} + \hbar \nabla \theta
\]

to cancel out the extra term

Putting together with \( \mathbf{p} \rightarrow \mathbf{p} - \frac{e}{c} \mathbf{A} \)

where \( \mathbf{A} \) is invariant to with \( \nabla \Lambda \) we see

that \( QM \) is gauge invariant if whenever we change gauge \( \mathbf{A} \rightarrow \mathbf{A} + \nabla \Lambda \) we also transform the wavefunction locally via

\[
\Psi(\mathbf{r}) \rightarrow e^{-ie/c \Lambda(\mathbf{r})} \Psi(\mathbf{r})
\]

\( \theta \leftrightarrow -\frac{e}{\hbar c} \Lambda \)
The abstract (and amazingly beautiful!)

way of looking at this calculation is:

If you insist on EM being locally gauge

invariant then you are forced to postulate the

existence of magnetic fields!!

An example of how symmetry \(\Rightarrow\) constraints

possible theories

Much more complex reasoning nowadays

If \(\otimes\) \(\Rightarrow\) Higgs particle etc.

must exist