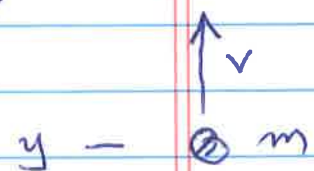
P ϕ A

Review Potential in Classical Mechanics

$\boxed{\text{First}}$ in $d=1$ and by example

$F=-mg$ $\downarrow$

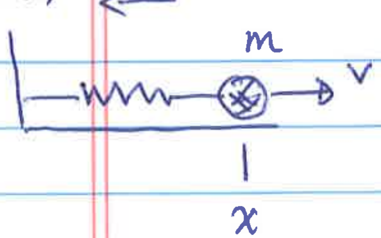


$$E = \frac{1}{2}mv^2 + mgy$$

$$\frac{dE}{dt} = \frac{1}{2}m\,2v\underbrace{a}_{-g} + mgv = 0$$

mass under influence of gravity

$F=-kx$ $\leftarrow$



$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$\frac{dE}{dt} = \frac{1}{2}m\,2v\underbrace{a}_{-\frac{kx}{m}} + \frac{1}{2}k\,2xv = 0$$

mass on spring

$\boxed{\text{Second}}$, in general

$$U(x) = -\int_0^x F\,dl$$

$$\frac{dU}{dt} = \frac{dx}{dt}\frac{dU}{dx}$$

chain rule!

recall Fundamental Thm of calculus discussion!

$$E = \frac{1}{2}mv^2 + U(x)$$

$$dE/dt = \frac{1}{2}m\,2v\underbrace{a}_{F(x)/m} - v\,F(x) = 0$$

P∅B RATIONALE TO GO THROUGH THIS $\nearrow$ a) good review
b) practice with vectors

$\boxed{\text{Finally}}$ in $d=3$ $\qquad v_x^2+v_y^2+v_z^2 = \vec{v}\cdot\vec{v}$

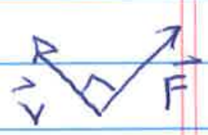
$$E = \tfrac{1}{2} m v^2 + U(\vec{r})$$

Dot product is Reasonable!

$$U(\vec{r}) = -\int_0^{\vec{r}} \vec{F}\cdot d\vec{\ell}$$

analogous to 1D

$$\frac{dU}{dt} = -\frac{d\vec{r}}{dt}\cdot\vec{F}$$



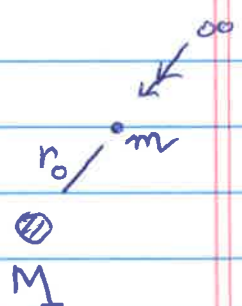
$$= -\frac{d\vec{r}}{dt}\cdot m\vec{a} = -\vec{v}\cdot m\vec{a}$$

$\vec{F}\cdot\vec{v} = 0$

no work done!

$$\frac{d}{dt}E = \tfrac{1}{2} m\, 2\vec{v}\cdot\vec{a} - \vec{v}\cdot m\vec{a} = \phi \quad !$$

KEPLER was big Example



$d\vec{\ell} = +\,dr\,\hat{r}$

coming in from ∞

$$U(\vec{r}_o) = -\int_0^{\vec{r}_o} -\frac{GMm}{r^2}\hat{r}\cdot d\vec{\ell}$$

"origin" is chosen at ∞

$$= +\int_{\infty}^{r_o} \frac{GMm}{r^2}\,dr$$

$$= -\frac{GMm}{r}\Big|_{\infty}^{r_o} = -\frac{GMm}{r_o}$$

$$E = \tfrac{1}{2} m v^2 - GMm/r$$

$$dE/dt = \phi$$

PϕC

What else is constant?

$$\vec{L} = \vec{r} \times \vec{p}$$

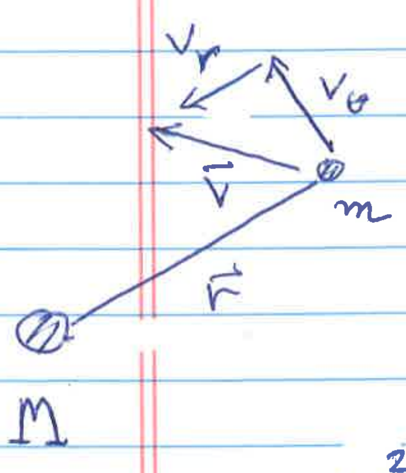
$$d\vec{L}/dt = d\vec{r}/dt \times \vec{p} + \vec{r} \times d\vec{p}/dt$$

$$= \underbrace{\vec{v} \times \vec{p}}_{\phi} + \underbrace{\vec{r} \times \vec{F}}_{\vec{\tau}\ \text{torque}}$$

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

true for any central Force $\longrightarrow$ $\vec{r} \times \vec{F} = 0$

$\vec{L} = \text{const} \equiv$ motion in plane $\leftarrow$ from direction of $\vec{L}$ fixed



$$\vec{v} = \underset{dx/dt}{v_x}\hat{x} + \underset{dy/dt}{v_y}\hat{y}$$

$$= \underset{dr/dt}{v_r}\hat{r} + \underset{r\,d\theta/dt}{v_\theta}\hat{\theta}$$

$$v^2 = v_x^2 + v_y^2 = v_r^2 + v_\theta^2$$

$$\vec{L} = m\vec{r} \times \vec{v}$$

$$|\vec{L}| = m|\vec{r}||\vec{v}|\sin\theta = m r v_\theta$$

PΦD

any conservative force

$\Downarrow$

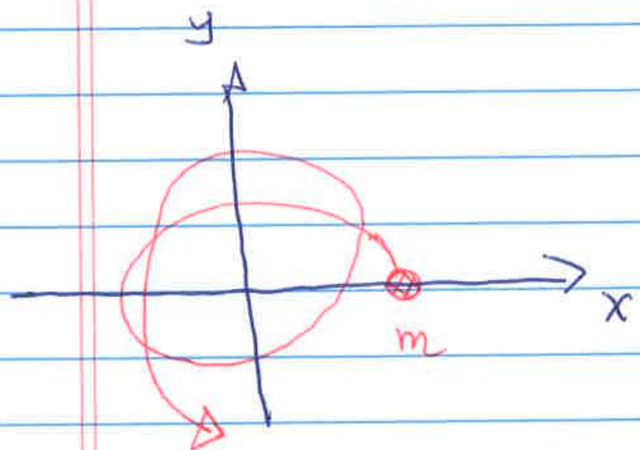
$E = \text{const}$

any central force

$\Rightarrow \vec{L} = \text{const} \begin{cases} |\vec{L}| = \text{const} \\ \text{direction } \vec{L} = \text{const} \end{cases}$

$\Downarrow$

motion in plane



so far...

nothing prevents this

unique to $1/r^2$ central force

and r central force

"closed orbits"

there is another conserved quantity!

Runge-Lenz vector!

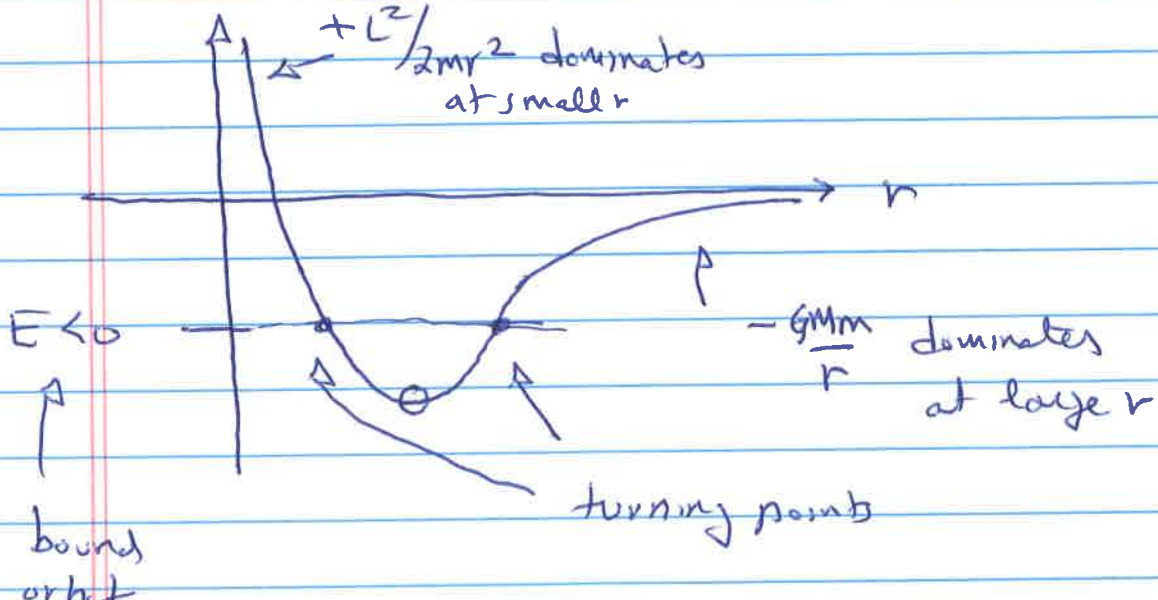
PϕE

$$E = \frac{1}{2} m v^2 - \frac{GMm}{r} = \text{const}$$

$$\frac{1}{2} m (v_r^2 + v_\theta^2)$$

$$\frac{1}{2} m\left(v_r^2 + \left(\frac{L}{mr}\right)^2\right) - \frac{GMm}{r} = E$$

$$\frac{1}{2} m\left(\frac{dr}{dt}\right)^2 + \underbrace{\frac{L^2}{2mr^2} - \frac{GMm}{r}}_{V_{eff}(r)} = E$$



circular orbit when? at minimum

$$0 = \frac{dV_{eff}}{dr} = \frac{-2L^2}{2mr^3} + \frac{GMm}{r^2}$$

$L = mvr$ $\qquad \frac{mv^2}{r} = \frac{GMm}{r^2}$ look familiar?!

$\vec{v} \perp \vec{r}$ for circular orbit

P6F

Why was this so useful?

At $\boxed{\text{no}}$ time did we have to solve for the actual trajectory of the planet around the sun.

This is actually a pretty hard problem, But even without doing it we know $\boxed{\text{a lot}}$

(2) $E < 0$ Bound orbits between turning point at r_{max} r_{min}

$$L^2/2mr^2 - GMm/r = E$$

(1) If $\frac{1}{2}mv_0^2 - \frac{GMm}{r_0} = E_0 > 0$ orbit unbound

ie Escape velocity is $v_0 = \left[2GM/r_0\right]^{1/2}$

(3) Circular orbit if $v_0^2/r_0 = \frac{GM}{r_0^2}$ ...