P1

Potentials (Griffiths Chapter 3)

Goal of electrostatics

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \rho(r')\, d\tau'$$

Easier

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r')\, d\tau'}{|\vec{r}-\vec{r}'|}$$

$$E = -\vec{\nabla} V$$

Still hard, so instead

$$\nabla^2 V = -\frac{1}{\epsilon_0} \rho \qquad \text{Poisson Eqn}$$

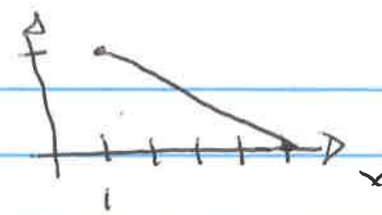
+ boundary conditions ← last lecture

Laplace Eqn $\nabla^2 V = 0$ is special case of $\rho = 0$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

P2

$V(1) = 4$, $V(5) = 0$ } BDY COND



$d = 1$ $\qquad \dfrac{d^2V}{dx^2} = 0$

$$V(x) = mx + b$$

(1) $V(x)$ is the average of $V(x+a)$ and $V(x-a)$ for any a

(2) No local maxima or minima

actually consequence of (1)

Extrema at bdys

Griffith's points out these are "silly" or "obvious" in $d=1$ BUT still ... in higher d.

$d = 2$ $\qquad \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$

(1) $\displaystyle V(x,y) = \frac{1}{2\pi R} \oint_{\text{circle}} V\,d\ell$ for any circle about (x,y)

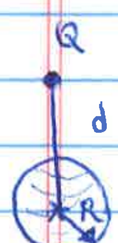
(2) V has no local maxima/minima

extrema at bdys

$$\underbrace{\frac{\partial^2 V}{\partial x^2} = -\frac{\partial^2 V}{\partial y^2}}_{\text{opposite sign}}$$

$\dfrac{\partial^2 f}{\partial x^2} > 0$

$\dfrac{\partial^2 f}{\partial x^2} < 0$



V at sphere center = average of V over surface

$d = 3$ $\qquad \displaystyle V(\vec{r}) = \frac{1}{4\pi R^2} \int_{\text{Sphere}} V\,da$ ← griffiths does several examples (at left, eg)

P2A

DISCRETE VERSION OF GRIFFITHS

You all know $\frac{dV}{dx} = \frac{V(x+dx)-V(x)}{dx}$.

What about $\frac{d^2V}{dx^2}$?

$$V(x+dx) = V(x) + V'(x)dx + \tfrac{1}{2}V''(x)dx^2 + \ldots$$

$$V(x-dx) = V(x) - V'(x)dx + \tfrac{1}{2}V''(x)dx^2 + \ldots$$

$$V''(x) = \frac{V(x+dx) - 2V(x) + V(x-dx)}{dx^2}$$

If $V''(x) = 0$

$$V(x) = \tfrac{1}{2}\left[V(x+dx) + V(x-dx)\right]$$

BASIS FOR NUMERICAL SOL'N OF LAPLACE EQN, ALSO CONNECTED TO (1) ON P2.

In 2D

$$V(x,y) = \tfrac{1}{4}\left\{V(x+dx,y) + V(x-dx,y) + V(x,y+dy) + V(x,y-dy)\right\}$$

Griffiths calls this "method of relaxation"

Very useful computationally

P3

Uniqueness Thm 1

Solution to Laplace Eqn in volume $\mathcal{V}$

is unique if V is given on surface $\mathcal{S}$ surrounding $\mathcal{V}$

proof $\quad \nabla^2 V_1 = 0 \qquad \nabla^2 V_2 = 0 \qquad$ Define $V_3 = V_2 - V_1$

$\Rightarrow \quad \nabla^2 V_3 = \nabla^2 V_2 - \nabla^2 V_1 = 0$ obeys Laplace

also $V_3 = V_2 - V_1 = 0$ on $\mathcal{S}$

But V_3 has no Local maxima and minima

$\Rightarrow V_3 = 0$

Uniqueness Thm 2

If $\mathcal{V}$ is surrounded by conductors and contains

a specified ρ, and the total charge on each conductor

is given $\Rightarrow \vec{E}$ is uniquely determined.

proof $\quad \vec{\nabla}\cdot\vec{E}_1 = \rho/\epsilon_0 \qquad \vec{\nabla}\cdot\vec{E}_2 = \rho/\epsilon_0$

$$\vec{E}_3 = \vec{E}_2 - \vec{E}_1 \quad \rightsquigarrow \quad \vec{\nabla}\cdot\vec{E}_3 = 0 \quad \rightsquigarrow \oint \vec{E}_3 \cdot d\vec{a} = 0$$

V_3 is constant on each conductor (charge distribution might be non trivial)

$$\vec{\nabla}\cdot V_3\vec{E}_3 = V_3 \underbrace{\vec{\nabla}\cdot\vec{E}_3}_{\phi} + \vec{E}_3\cdot \underbrace{\vec{\nabla}V_3}_{-\vec{E}_3} = -E_3^2$$

P4

$$\int_V \vec{\nabla}\cdot V_3\vec{E}_3 \, d\tau = \oint V_3 \vec{E}_3 \cdot d\vec{a} = -\int_V E_3^2 \, d\tau$$

Constant

but $\int \vec{E}_3 \cdot d\vec{a} = 0$ so $\emptyset$

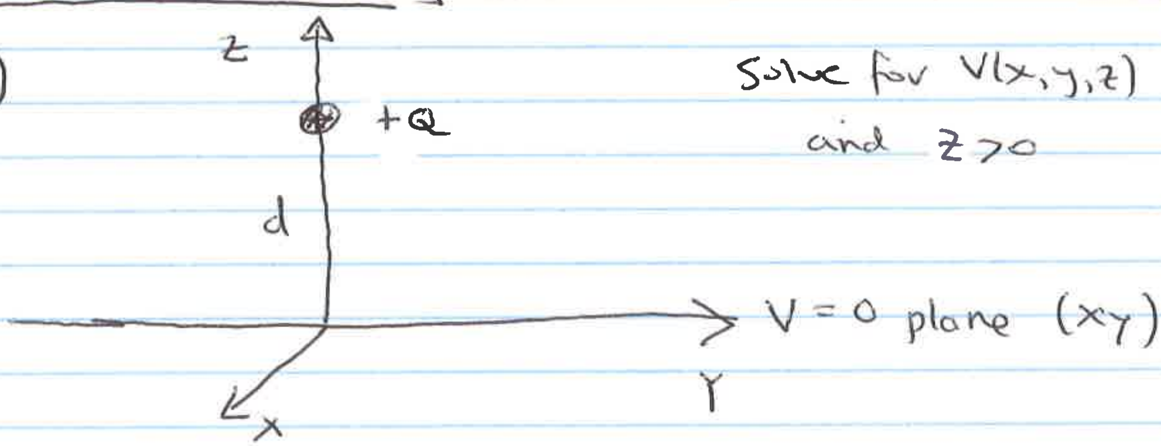
so $\vec{E}_3 = 0$

Thus $\vec{E}_1 = \vec{E}_2$.

* If you can cleverly guess $\boxed{\text{a}}$ solution, the uniqueness theorems guarantee you have found $\boxed{\text{the}}$ solution.

$\boxed{\text{METHOD OF IMAGES}}$

(1)



Solve for $V(x,y,z)$ and $z > 0$

Image charge $-Q$ at $(0,0,-d)$,

Clearly $V(x,y,0) = 0$ $\quad$ * $V(xyz) = \frac{Q}{4\pi\epsilon_0}\left\{ \frac{1}{[x^2+y^2+(z-d)^2]^{1/2}} - \frac{1}{[x^2+y^2+(z+d)^2]^{1/2}} \right\}$

The only charge in $z > 0$ is the original charge $+Q$

so * is the solution

As long as image charges are <u>outside</u> region you are okay since not altering ρ.

P5

Induced surface charge on plane.

$$\sigma = -\epsilon_0 \, \partial V / \partial n$$

$$\frac{\partial V}{\partial z} = \frac{Q}{4\pi\epsilon_0}\left\{ \frac{-(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}} \right\}$$

on $z=0$ surface $\qquad \sigma(x,y) = -Qd \Big/ 2\pi\left[x^2+y^2+d^2\right]^{3/2}$

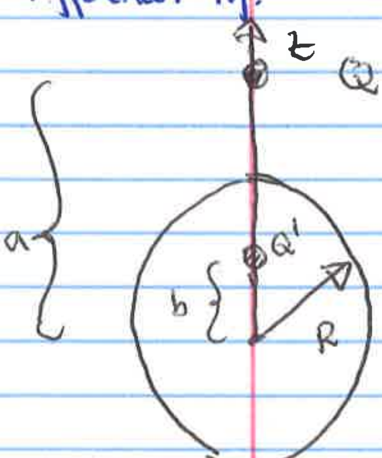
$\sigma(x,y) < 0$ as expected

$Q = \int \sigma \, da = -Q$ easy to show

Force on $+Q$ from image $\quad -\frac{1}{4\pi\epsilon_0} \frac{Q^2}{(2d)^2} \hat{z}$

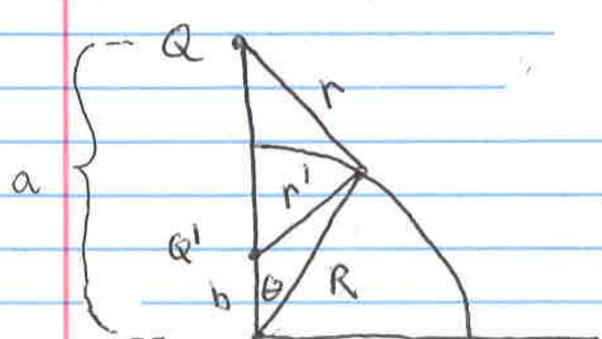
Griffiths: Compute energy in field and compare to 2 pt charges. Factor of 2 difference, why?

(2) Point charge Q near grounded conducting sphere



Put point charge Q' at distance b from origin and see if you can adjust Q' and b to make $V=0$ on surface of sphere

If $Q' = -R/a \, Q$ $\quad b = R^2/a$ then $V=0$ on sphere



$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{Q'}{r'}\right)$$

P5A

$$\frac{Q'}{r'} = -\frac{R}{a} Q \frac{1}{(R^2+b^2-2Rb\cos\theta)^{1/2}}$$

$$= -\frac{R}{a} Q \left[R^2 + \frac{R^4}{a^2} - \frac{2R^3}{a}\cos\theta \right]^{-1/2}$$

$$= -RQ \left[R^4 + R^2a^2 - 2R^3 a\cos\theta \right]^{-1/2}$$

But $\quad R^2 + a^2 - 2Ra\cos\theta = r^2$

$$= -RQ\frac{1}{R} \left[R^2 + a^2 - 2Ra\cos\theta \right]^{-1/2}$$

$$= -Q/r. \qquad \checkmark\checkmark$$

parabola
circle
ellipse
hyperbola

Apollonius 200BC "locus of all points a fixed ratio of distances from two given points is sphere"

Kelvin 1848 solved at age 24

Separation of Variables in Cartesian Coordinates

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

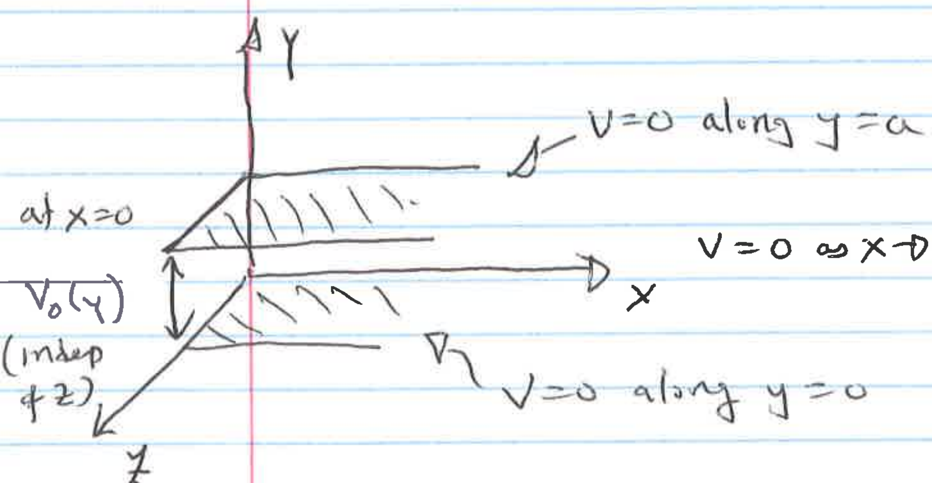
$$V(x,y) = f(x)g(y)$$

$$g(y)\frac{d^2}{dx^2}f = -f(x)\frac{d^2 g}{dy^2}$$

$$\frac{1}{f(x)}\frac{d^2 f}{dx^2} = \frac{-1}{g(y)}\frac{d^2 g}{dy^2} = \text{Const} \equiv k^2$$

$$\left.\begin{aligned} f(x) &= e^{kx},\ e^{-kx} \\ g(y) &= \sin ky,\ \cos ky \end{aligned}\right\} \quad V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

Boundary conditions



$V = 0$ as $x \to \infty$ $\rightsquigarrow$ $A = 0$

$V=0$ along $y=0$ $\rightsquigarrow$ $D = 0$

$$V(x,y) = Ae^{-kx}\sin ky$$

can depend on k

$V = 0$ at $y = a$

$\Downarrow$

$$k = \frac{n\pi}{a}$$

P7

$$V(x,y) = \sum_n A_n e^{-m\pi x/a} \sin\frac{n\pi y}{a}$$

Superposition

$$V_0(y) = \sum_n A_n \sin\frac{n\pi y}{a}$$

$$\int_0^a \sin\frac{n\pi y}{a} \sin\frac{m\pi y}{a}\, dy = \frac{a}{2}\delta_{nm}$$

$$\Rightarrow \quad A_m = \frac{2}{a}\int_0^a V_0(y) \sin\frac{m\pi y}{a}\, dy$$

Vector analog

$\vec{v} = \sum a_n \hat{e}_n$

$\hat{e}_m \cdot \vec{v} = a_m$

Since $\hat{e}_n \cdot \hat{e}_m = \delta_{nm}$

$$V(x,y) = \sum_{m=1}^{\infty} \frac{2}{a}\int_0^a V_0(y) \sin\frac{m\pi y}{a}\, dy\, e^{-m\pi x/a}$$

$$= \int_0^a \underbrace{\left[\frac{2}{a}\sum_{m=1}^{\infty} \sin\frac{m\pi y}{a} e^{-m\pi x/a}\right]}_{G(xy)} V_0(y)\, dy$$

$G(xy)$ ← "Greens function" or "propagator"

$$V(x,y) = \int_0^a G(xy) V_0(y)\, dy$$

G depends on PDE and Bdy conditions!

propagates soln $V_0(y)$ at $x=0$ to non-zero x

If $V_0(y) = V_0$, a constant

$$A_m = \frac{2}{a}\int_0^a \sin\frac{m\pi y}{a}\,dy\,V_0$$

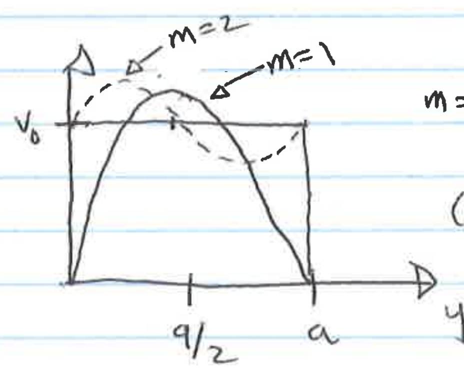
$$= \frac{2V_0}{a}\frac{a}{m\pi}\left.-\cos\frac{m\pi y}{a}\right|_0^a = \frac{-2}{m\pi}\Big[\underbrace{\cos m\pi}_{(-1)^m} - 1\Big]V_0$$

$$A_m = +\frac{4V_0}{m\pi} \qquad m = 1, 3, 5, \ldots$$

$$A_m = 0 \qquad m = 2, 4, 6 \ldots$$

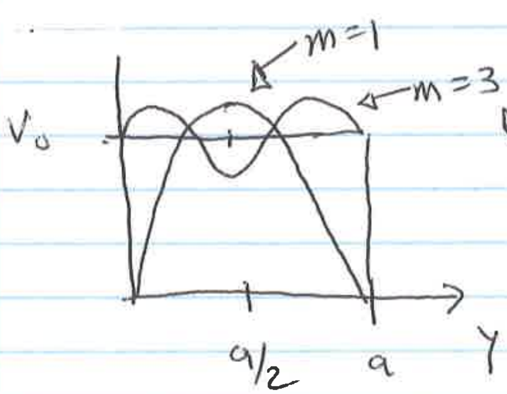
$$V(x\,y) = \sum_{m\,odd} \frac{4V_0}{m\pi} e^{-m\pi x/a} \sin\frac{m\pi y}{a}$$

$m = 1$ $\quad \frac{4}{\pi}\sin \pi y/a$



$m = 2$ would be bad! (asymmetric)

$\Rightarrow a_2 = 0$



reduces V at $y = a/2$
increases V at $y = a/4, 3a/4$

This is, of course, an example of Fourier series