P1

$\boxed{\text{POTENTIAL}}$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} = \frac{Q}{4\pi\epsilon_0}\frac{\overbrace{x\hat{x} + y\hat{y} + z\hat{z}}^{\vec{r}}}{\underbrace{(x^2+y^2+z^2)^{3/2}}_{r^3}}$$

$$\left(= \frac{Q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} \right)$$

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix}$$

$$r = (x^2+y^2+z^2)^{1/2}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}2y(x^2+y^2+z^2)^{-\frac{1}{2}} = y/r$$

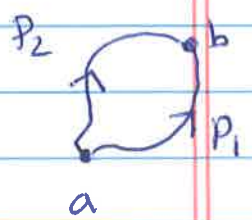
$$(\vec{\nabla}\times\vec{E})_z = \frac{\partial}{\partial y}\frac{z}{r^3} - \frac{\partial}{\partial z}\frac{y}{r^3}$$

$$= -\frac{3z}{r^4}\,y/r - \left(\frac{-3y}{r^4}\frac{z}{r}\right) = 0$$

$$\oint_P \vec{E}\cdot d\vec{l} = \int_S \vec{\nabla}\times\vec{E}\cdot d\vec{a} = 0$$

$$\Downarrow$$

$\int_a^b \vec{E}\cdot d\vec{l}$ is independent of path



Define the potential

if $\int_{P_1} \vec{E}\cdot d\vec{l} \neq \int_{P_2} \vec{E}\cdot d\vec{l}$

$$V(r) = -\int_0^r \vec{E}\cdot d\vec{l}$$

↑ some origin

$$\int_{P_1} - \int_{P_2} \neq 0$$

$$\oint_P \vec{E}\cdot d\vec{l} \neq 0$$

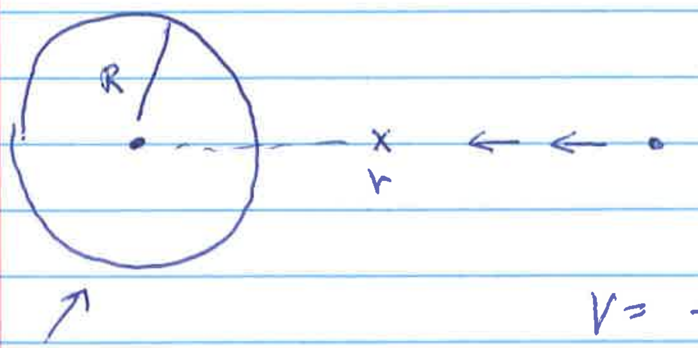
potential Energy

$$U(r) = QV(r) = -\int_0^r \vec{F}\cdot d\vec{l}$$

P2

Because $\vec{F}$ and $\vec{E}$ obey superposition, so does the potential : $V = V_1 + V_2 + V_3 + \dots$

individual potentials due to $q_1, q_2, q_3, \dots$



$r > R$

shell charge
Q

$$V = -\frac{1}{4\pi\epsilon_0}\int_\infty^r Q/r'^2 \, dr'$$

$$= \frac{1}{4\pi\epsilon_0} Q/r' \Big|_\infty^r$$

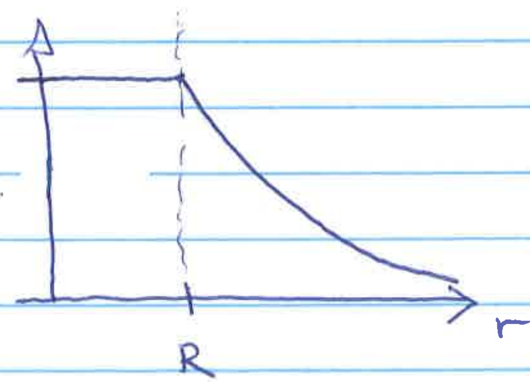
$$= Q/4\pi\epsilon_0 r$$

$r < R$

E = 0 inside

$$V = -\frac{1}{4\pi\epsilon_0}\int_\infty^R \frac{Q}{r'^2} dr' - \frac{1}{4\pi\epsilon_0}\int_R^r 0 \, dr'$$

$$= Q/4\pi\epsilon_0 R$$



P 3

$\boxed{\text{Laplace's Eqn}}$

$$E = -\nabla V$$

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \qquad \bar{\nabla} \times \bar{E} = 0$$

$$\boxed{\nabla^2 V = -\rho/\epsilon_0}$$ Poisson Eqn

if $\rho = 0$

$$\boxed{\nabla^2 V = 0}$$ Laplace Eqn

We will spend several weeks solving these eqns. All of Chapter 3 of Griffiths.

Why Impt ← central to EM but also

time dependent

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r,t) + V(r)\psi(r,t) = i\hbar\frac{\partial}{\partial t}\psi(r,t)$$

$\psi(r,t) = \phi(r)\, e^{-iEt/\hbar}$

time independent

$$-\frac{\hbar^2}{2m}\nabla^2\phi + V(r)\phi(r) = E\phi(r)$$

Schroedinger Eqn of QM

Diffusion Eqn of Classical Mechanics

$$D\nabla^2\psi(r,t) = \frac{\partial\psi(r,t)}{\partial t}$$

Schroedinger $V(r) = 0$

etc

$$\frac{i\hbar}{2m}\nabla^2\psi(r,t) = \frac{\partial\psi(r,t)}{\partial t}$$

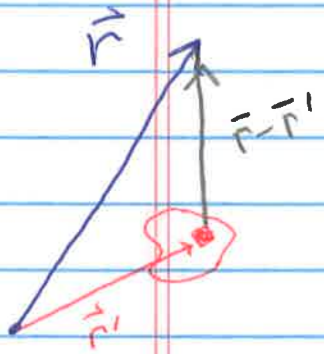
"imaginary time diffusion Eqn"

similar physics

"spreading of wavepacket"

P4

$$V(r) = \frac{1}{4\pi\epsilon_0} \sum q_i / |\bar{r} - \bar{r}_i|$$ Set of discrete point charges



$$\frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho(r')\, d\tau'}{|r - r'|}$$ Continuous volume charge distribution

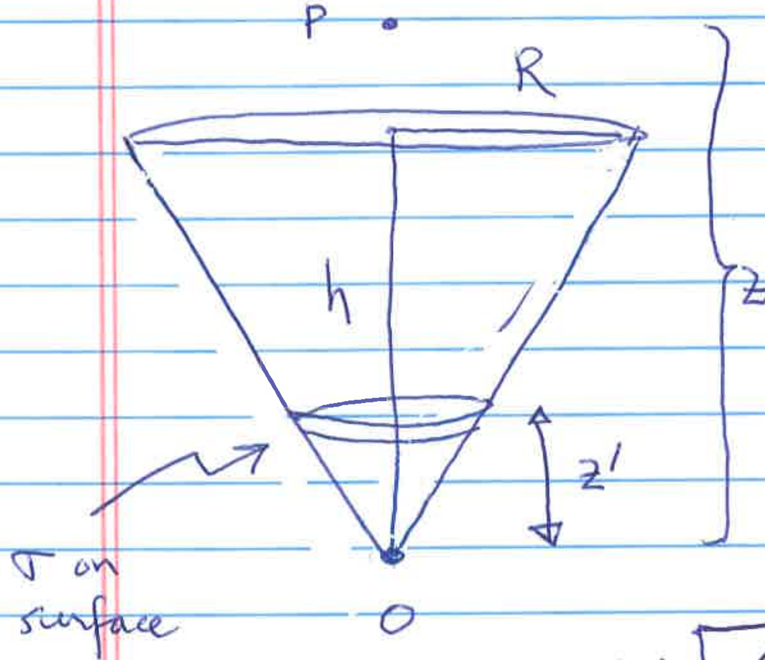
$$\frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma(r')\, da'}{|r - r'|}$$ surface

$$\frac{1}{4\pi\epsilon_0} \int_P \frac{\lambda(r')\, dl'}{|r - r'|}$$ linear

p5

Griffiths 2-26

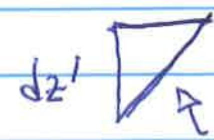
Potential difference between base and apex of cone



Divide cone into rings

$$da' = 2\pi \frac{Rz'}{h} \frac{\sqrt{R^2+h^2}}{h} dz'$$

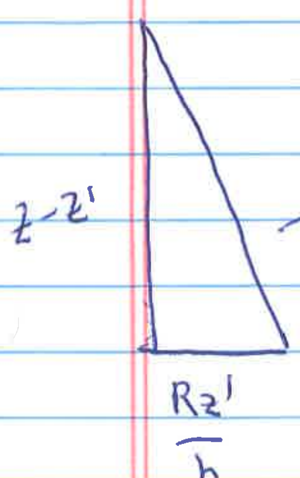
ring radius



$$\frac{\sqrt{R^2+h^2}}{h} dz'$$

potential of such a ring at position z

$$\frac{1}{4\pi\epsilon_0} \sigma da' \Big/ \left[(z-z')^2 + (Rz'/h)^2\right]^{1/2}$$



NB: Griffiths asks about R=h case

P6

potential at base of cone $z=h$

$$V_{base} = \int_0^h \frac{\sigma}{4\pi\epsilon_0} \frac{2\pi R z'}{h} \frac{\sqrt{R^2+h^2}}{h} dz' \frac{1}{\sqrt{(h-z')^2+\left(\frac{Rz'}{h}\right)^2}}$$

$z=h$ at base

V_{apex} is same formula except with $z=0$

V_{apex} is easier integral. Let's do it first

$$V_{apex} = \frac{\sigma R\sqrt{R^2+h^2}}{2h^2\epsilon_0} \int_0^h \frac{z'dz'}{\left[z'^2(1+R^2/h^2)\right]^{1/2}}$$

$$\frac{z'}{h}(h^2+R^2)^{1/2}$$

$$= \frac{\sigma R}{2h\epsilon_0} \underbrace{\int_0^h dz'}_{h} = \frac{\sigma R}{2\epsilon_0} \left(= \frac{\sigma h}{2\epsilon_0} \text{ if } R=h \right)$$

check units

$$[\sigma] = \frac{Q}{L^2} \qquad [\sigma R] = \frac{Q}{L} \quad ✓$$

P7

Let's do Griffiths $R = h$ for simplicity

$$V_{base} = \frac{\sigma R \sqrt{R^2+h^2}}{2h^2 \epsilon_0} \int_0^h \frac{z' \, dz'}{(2z'^2 - 2hz' + h^2)^{1/2}}$$

$$(h-z')^2 + z'^2$$

add and subtract

$$\frac{z'}{(2z'^2 - 2hz' + h^2)^{1/2}} = \frac{\frac{1}{4}(4z' - 2h) + \frac{1}{2}h}{(2z'^2 - 2hz' + h^2)^{1/2}}$$

derivative gives $4z' - 2h$

First integral is then

$$\frac{1}{2}(2z'^2 - 2hz' + h^2)^{1/2} \Big|_0^h$$

$$= \frac{1}{2} h$$

Second integral harder

$$2z'^2 - 2hz' + h^2 = 2\left(z'^2 - hz' + \frac{h^2}{4}\right) + \frac{h^2}{2}$$

$$= 2(z' - h/2)^2 + h^2/2$$

$$z' - h/2 = \frac{h}{2} \tan\theta \qquad dz' = \frac{h}{2} \sec^2\theta \, d\theta$$

P9A

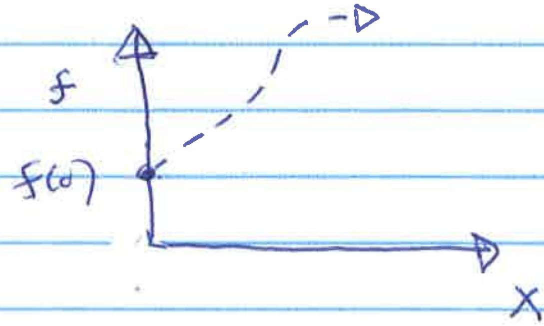
Aaron Little 86 HW2

Zack Tyler ? check score

General Concepts of role of boundaries in solution of differential eqns:

Because often $t=0 \Rightarrow$ "initial conditions"

$\boxed{\text{ODE}}$



First order

① $\frac{df}{dx} = x+3$

$f(x) = \frac{1}{2}x^2 + 3x + c$

$f(0) = c$ ← need initial value

or, actually, value anywhere

$f(1) = \frac{1}{2} + 3 + c$

$c = f(1) - 7/2$

③ Second order

$$\frac{d^2 f}{dx^2} = -4f$$

$f(x) = A\sin 2x + B\cos 2x$

can you specify $f(0)$ and $f'(0)$?

Yes $f(0) = B$

$f'(x) = 2A\cos 2x - 2B\sin 2x$

$f'(0) = 2A$

What about $f(0)=0$ $A=0$

and $f(\pi)=0$

$f(x) = B\cos 2x$

$f(\pi) = B\cos 2\pi$

Only $B=0 \Rightarrow$ no nontrivial soln!

Moral: Some Boundary Conditions do not allow sol'n to Differential Eqn!

② $\frac{d^2x}{dt^2} = a$ ← constant

$x(t) = x(0) + v(0)t + \frac{1}{2}at^2$

could you specify $x(0)$ and $x(10)$ instead?

$x(10) = x(0) + c\,10 + \frac{1}{2}a\,100$

can always solve for c

P9B

$$\frac{d^2 f}{dx^2} = -k^2 f \qquad A\sin kx + B\cos kx$$

$$f(0) = 0 \quad \rightarrow \quad A = 0$$

$$f(L) = 0 \quad \rightarrow \quad B\cos kL = 0$$

What can you say about k?

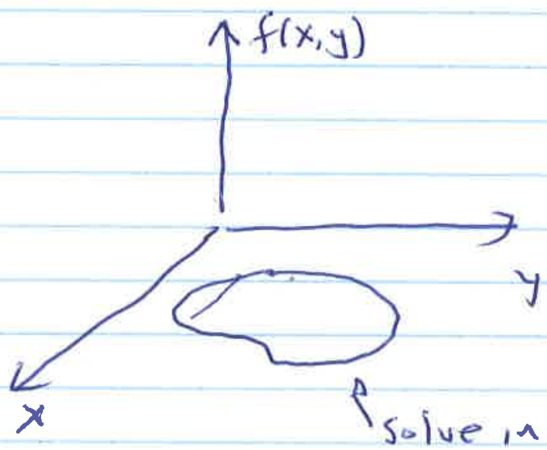
Where have you seen this problem? Schroedinger Eqn particles in a box

k related to E $\rightarrow$ Energy quantization

PDE Much richer

Poisson:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\rho(x,y)}{\epsilon_0}$$



Solve in some region of space

What Boundary conditions

f on bdy?

Boundary Conditions

In classical Mechanics

$F = ma = m\frac{d^2x}{dt^2}$ not enough need x_0 v_0 to get $x(t)$

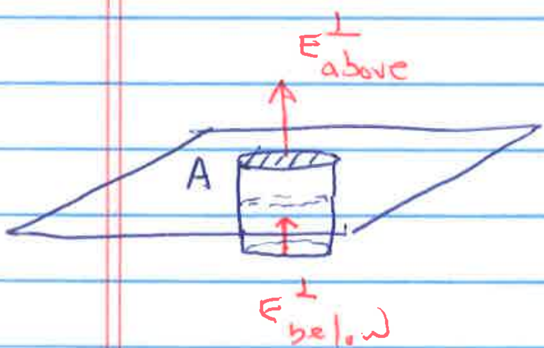
2nd order

2 numbers

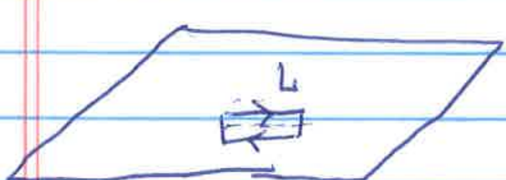
$\vec{E}$ or V on boundary need to be combined with

$$\nabla^2 V = \rho/\epsilon_0$$

$\Rightarrow$ what happens to $\vec{E}$, V across boundary?



$$\left(E^{\perp}_{above} - E^{\perp}_{below}\right)A = \frac{\sigma A}{\epsilon_0}$$



$$\left(E^{\parallel}_{above} - E^{\parallel}_{below}\right)L = 0$$

P10

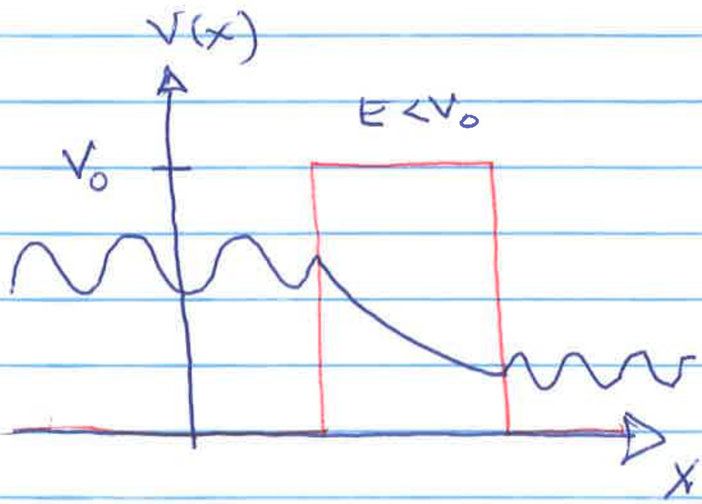
Potential is continuous $V_{above} = V_{below}$

But $\frac{\partial V}{\partial n} = \vec{\nabla} V \cdot \hat{n}$

$$\left(\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} \right) = -\sigma/\epsilon_0$$

rephrasing of $E^{\perp}_{above} - E^{\perp}_{below} = \sigma/\epsilon_0$

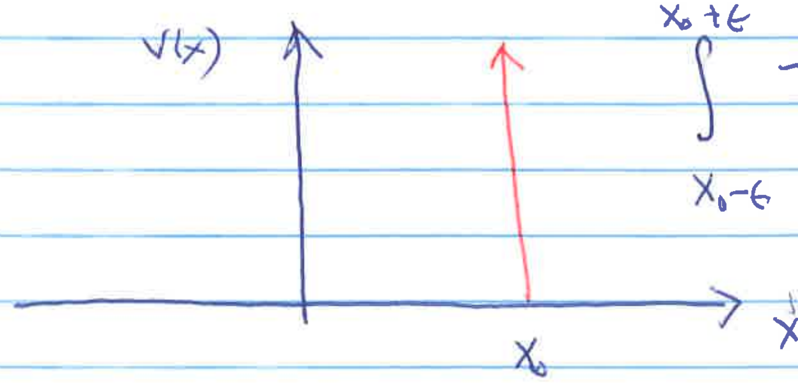
In QM



$\psi(x)$ is continuous

as is $\frac{d\psi}{dx}$

Unless $V(x)$ is delta function $V_0 \delta(x - x_0)$



$$\int_{x_0-\epsilon}^{x_0+\epsilon} -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + V(x)\psi(x) = E\psi(x)$$

$$-\frac{\hbar^2}{2m} \left. \frac{d\psi}{dx} \right|_{x_0-\epsilon}^{x_0+\epsilon} + V_0 \psi(x_0) = 0$$

Work to move charge from $\vec{a}$ to $\vec{b}$

$$W = Q\left(V(\vec{b}) - V(\vec{a})\right)$$

Energy of collection of point charges

$$W_2 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}}$$

start with q_1 and bring up q_2

$$W_3 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

bring up q_3

$$W_{TOT} = \frac{1}{4\pi\epsilon_0} \sum_{i>j} q_i q_j / r_{ij}$$

$$= \frac{1}{2} \sum_j q_j \underbrace{\sum_{i\neq j} q_i / r_{ij} \; 1/4\pi\epsilon_0}_{V(r_j)}$$

$$= \frac{1}{2}\sum_j q_j V(r_j)$$

P14

Conductors ← charges free to move (definition)

(1) $\vec{E} = 0$ inside (otherwise charges will move!)

(2) $\rho = 0$ inside since $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$

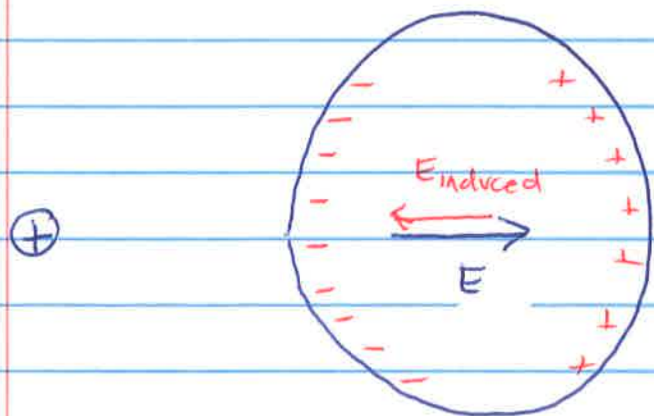
(3) All charge on surface (consequence of (2))

(4) Entire conductor is equipotential (consequence of (1))

(5) $\vec{E} \perp$ to surface outside conductor (otherwise charge will move)

P15

Induced charges

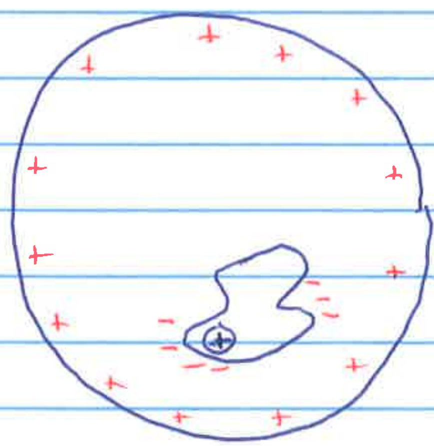


Conductor

(1) free to move charges will rearrange so − nearest ⊕ and + farthest away

(2) must rearrange to get $\vec{E} = 0$ inside

Pretty fantastic stuff



unchanged conductor with weird hole containing ⊕ Q somewhere

$E = \frac{1}{4\pi\epsilon_0} Q/r^2 \, \hat{r}$ outside!!

no matter what shape of hole is and where ⊕ Q is inside hole

⊕ Q induces ⊖ Q on cavity walls ⇒ ⊕ Q on outer surface

This last +Q is uniformly distributed!

P16

Capacitors

two conductors carrying opposite charge

$E \sim Q$

$\Rightarrow V \sim Q$

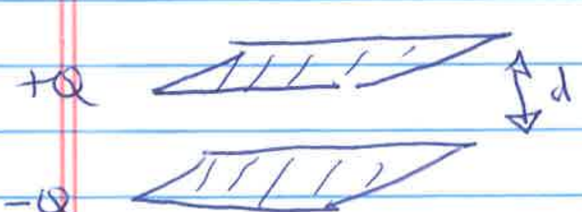
potential difference between conductors

Constant of proportionality $C = Q/V$ $\quad V = Q/C$

If C large can carry big charge Q without a large V

useful in electronic device

high V ← bad (breakdown etc)



Simplest geometry

$E = \sigma/\epsilon_0 \qquad V = \frac{\sigma d}{\epsilon_0}$

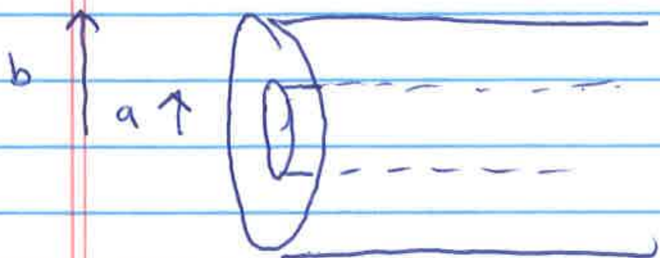
$Q = \sigma A$

$$C = \frac{\sigma A}{\sigma d/\epsilon_0} = \frac{\epsilon_0 A}{d}$$

only depends on geometry

P17

Coaxial cable



Gauss law

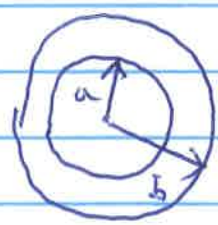
$$\frac{\lambda \ell}{\epsilon_0} = 2\pi r \ell E$$

$$E = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}$$

$$V = \int_a^b \frac{\lambda}{2\pi\epsilon_0}\frac{dr}{r} = \frac{\lambda}{2\pi\epsilon_0} \ln b/a$$

$$C = \frac{Q}{V} = \frac{\lambda L}{\frac{\lambda}{2\pi\epsilon_0}\ln b/a} = \frac{2\pi\epsilon_0 L}{\ln(b/a)}$$

Concentric Spherical Shells



$$V = \int_a^b \frac{Q}{4\pi\epsilon_0}\frac{dr}{r^2}$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{Q}{4\pi\epsilon_0}\frac{b-a}{ab}$$

$$C = \frac{Q}{V} = \frac{4\pi\epsilon_0\, ab}{b-a} \rightarrow 4\pi\epsilon_0 \frac{a^2}{d} = \frac{\epsilon_0 A}{d}$$

$b = a + d$ ← small