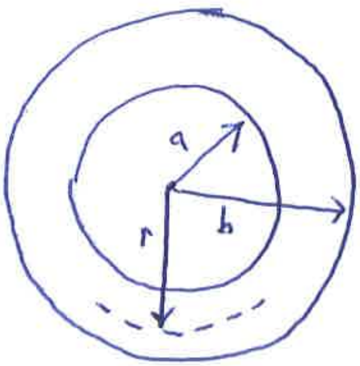
PHYSICS 110A, WINTER 2018
ELECTRICITY AND MAGNETISM

Second Midterm Exam

[**1.**] Two infinitely long cylindrical conductors have a common axis and radii a and b. What is the capacitance of this arrangement? Note: You *must* give a complete derivation of all equations used. For example, if you use a formula for the electric field E, show clearly where it comes from. Take the limit $b \to a + \Delta$, where Δ is small, and relate your answer to the formula $C = \epsilon_0 A/d$ for two large conducting planes with area A and separation d. Possibly useful hint: $\ln(1+x) \sim x$ for x small.



Draw imaginary Gaussian surface of cylindrical shape at radius r. Gauss' Law gives

$$\Phi_E = 2\pi r E L = 1/\epsilon_0 \, \lambda L$$

λ = charge/length on inner conductor

Φ_E: all flux through cylinder sides since $\vec{E} \perp$ to end caps

L: cylinder length

$$\Rightarrow E = \frac{1}{2\pi\epsilon_0 r}\lambda$$

The potential at "a" is higher than at "b" by

$$V_{ba} = -\int_b^a \underbrace{E\,dr}_{\vec{E} \parallel \hat{r}} = -\frac{1}{2\pi\epsilon_0}\lambda \ln r \Big|_b^a = \frac{\lambda}{2\pi\epsilon_0}\ln b/a$$

$$C = Q/V = \lambda L \Big/ \frac{\lambda}{2\pi\epsilon_0}\ln b/a = 2\pi\epsilon_0 L / \ln b/a$$

If $b = a + \Delta$ and Δ is small $\quad \ln b/a = \ln(1 + \Delta/a) \approx \Delta/a$

$$C \approx \epsilon_0 \underbrace{2\pi a L}_{\text{Area } A \text{ of cylinder}} \frac{1}{\Delta} \rightarrow \epsilon_0 A/d$$

Δ: distance d between cylinders

[2.] Find the general solution to Laplace's equation in spherical coordinates for the case when V depends only on r.

The Laplacian in spherical coordinates is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\phi^2})$$

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial V}{\partial r} = 0$$

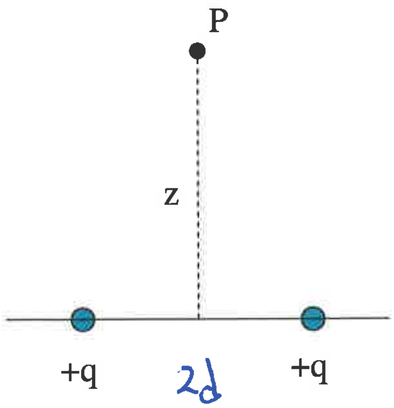
since V is independent of θ, ϕ

$$\frac{\partial}{\partial r} r^2 \frac{\partial V}{\partial r} = 0$$

$$r^2 \frac{\partial V}{\partial r} = A$$

$$\frac{\partial V}{\partial r} = A/r^2$$

$$V(r) = B - A/r$$



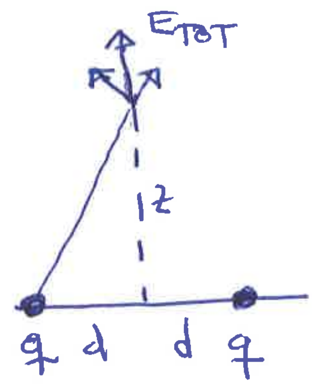
[3.] Find the potential at a distance z above the center of the charge distribution in the Figure. Compute $\mathbf{E} = -\nabla V$. Show your result agrees with a direct calculation of $\mathbf{E}$ from Coulomb's Law. Suppose the right hand charge is changed to $-q$. What is the potential at P? What field does that suggest? What is the actual field? Explain carefully any discrepancy.

$$V = 2\frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{z^2+d^2}} = \frac{2q}{4\pi\varepsilon_0}(z^2+d^2)^{-1/2}$$

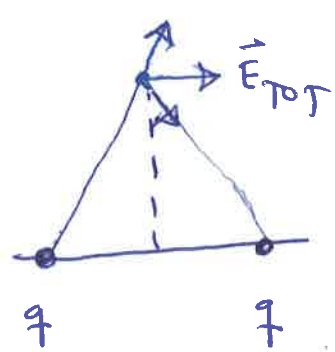
$$\vec{E} = -\vec{\nabla}V = -\hat{z}\frac{\partial V}{\partial z} = \frac{-2q}{4\pi\varepsilon_0}(z^2+d^2)^{-3/2}(-1/2)(2z)$$

$$= 2\underbrace{\frac{q}{4\pi\varepsilon_0}\frac{1}{z^2+d^2}\frac{z}{(z^2+d^2)^{1/2}}}_{\text{z component of } \vec{E} \text{ field due to one of charges}}\hat{z}$$

components in other directions cancel



If the right hand charge is $-q$ we get $V=0$.
This suggests that $\vec{E} = -\vec{\nabla}V = 0$. But $\vec{E} \neq 0$, it is in $\hat{x}$ direction. To get correct $\vec{E}$ we need $V(x,y,z)$ so that we can get $\frac{\partial V}{\partial x}$
we cannot just derive V at $(0,0,z)$



[4.] A sphere of radius R carries a charge density $\rho(r) = Ar$ where A is a constant. Find the energy of the configuration. (There are several ways to do it!) Derive any equations, *e.g.* for the electric field, that you use.

Outside the sphere Gauss Law gives $4\pi r^2 E = \frac{Q}{\epsilon_0}$

where we get charge Q by integrating ρ

$$Q = \int_0^R 4\pi r^2 dr\, Ar = 4\pi A \frac{r^4}{4}\Big|_0^R = \pi A R^4$$

inside the sphere

$$4\pi r^2 E = \frac{1}{\epsilon_0}\int_0^r 4\pi r'^2 dr' A r' = \frac{4\pi}{\epsilon_0}\frac{r^4}{4} A$$

$$E = \frac{A}{4\epsilon_0} r^2$$

One way to get the Energy is to integrate $\frac{1}{2}\epsilon_0 E^2$ over all space

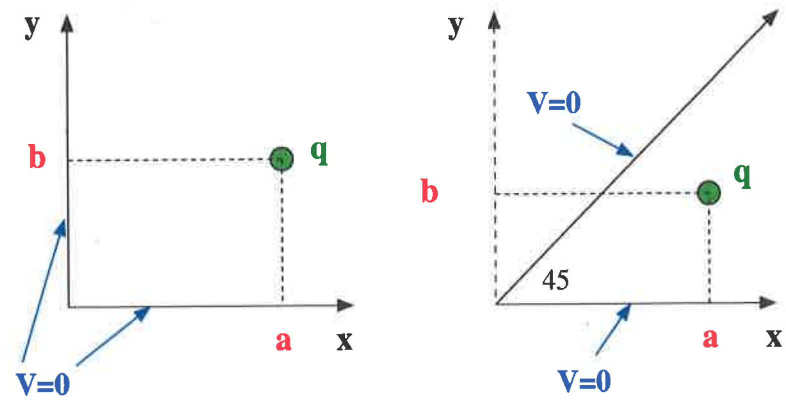
$$\text{Energy} = \underbrace{\int_0^R \left(\frac{Ar^2}{4\epsilon_0}\right)^2 \frac{1}{2}\epsilon_0\, 4\pi r^2 dr}_{\text{Inside}} + \underbrace{\int_R^\infty \left(\frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\right)^2 4\pi r^2 dr \frac{1}{2}\epsilon_0}_{\text{outside}}$$

$$= \frac{A^2\pi}{8\epsilon_0}\underbrace{\int_0^R r^6 dr}_{R^7/7} + \underbrace{(\pi A R^4)^2}_{Q} \frac{1}{8\pi\epsilon_0}\underbrace{\int_R^\infty \frac{1}{r^2} dr}_{1/R}$$

$$= \frac{A^2R^7\pi}{8\epsilon_0}\left\{\frac{1}{7} + 1\right\} = \frac{A^2R^7\pi}{7\epsilon_0}$$

Check units:
$[A] = Q/L^4$
so $[A^2R^7]$
$= Q^2/L$
which is correct!

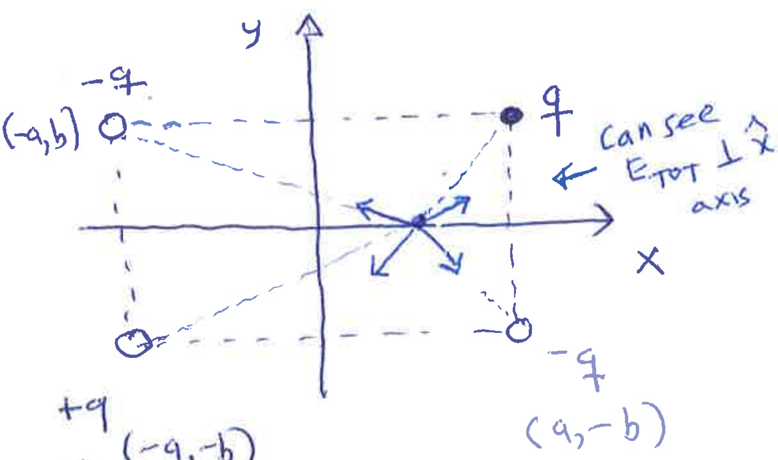
You can also get the same answer from $\int \rho V\, d\vartheta$ but it is harder because you must get the potential V from E.



Explain your reasoning,

[5.] Two semi-infinite grounded conducting planes meet at right angles. In the region between them, there is a point charge q situated as in the left side of the Figure above. Set up the image configuration, and calculate the potential in the region. What charges do you need and where should they be located? ~~What is the force on q?~~ Can you do the problem if the planes meet at 45°, as in the right side of the Figure? You do not need to compute the force in this case.

From problem with a single plane we need image charges $-q$ at $(a,-b)$ and $(-a,b)$.

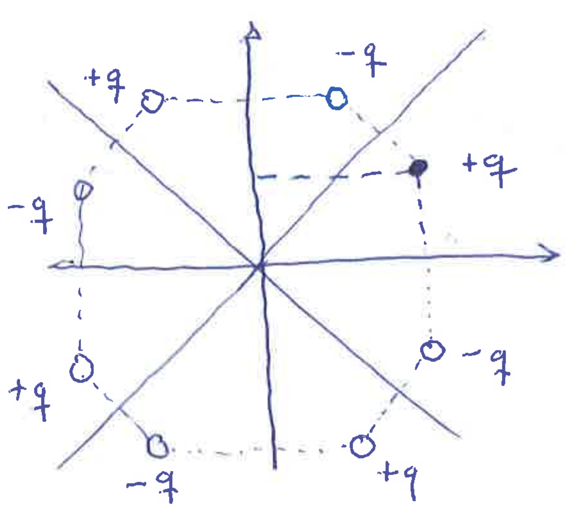


But clearly this is not quite enough because their fields will not be ⊥ to x, y axes when combined with that of original charge, need one more image $+q$ $(-a,-b)$!

!!

$$V(x,y) = \frac{q}{4\pi\epsilon_0}\left\{\left[(x-a)^2+(y-b)^2\right]^{-1/2} - \left[(x-a)^2+(y+b)^2\right]^{-1/2} + \left[(x+a)^2+(y+b)^2\right]^{-1/2} - \left[(x+a)^2+(y-b)^2\right]^{-1/2}\right\}$$

You can also do the 45° case:



Can you see that the only angles that work are ones that have the collection of images start repeating after you go around a complete 360°?

Those magic angles are ones which divide 360° an even # of times: 180°, 90°, 60°, 45°

2, 4, 6, 8