VI

$\boxed{\text{Why vectors?}}$ ← Griffiths starts his book with 30 pages on vectors

Central to our subject are Electromagnetic "fields": At each location in (3D) space $\vec{r}$ there is a vector $\vec{E}(\vec{r})$. We will see that a charge q experiences an <u>electric</u> <u>force</u>

$$\vec{F} = q\vec{E}(\vec{r})$$

if it is located at $\vec{r}$.

Likewise, at each point in space there is a vector $\vec{B}(\vec{r})$. We will see that a charge q moving with velocity $\vec{v}$ while located at position $\vec{r}$ feels a magnetic force

$$\vec{F} = q\vec{v} \times \vec{B}(r)$$

$\boxed{Q}$ (Interestingly q also is the <u>source</u> of $\vec{E}$ and $\vec{B}$ besides responding to them!)

V2

So... vectors appear all over the place

classical mechanics $\vec{r}, \vec{v}, \vec{F}$, $\vec{E}, \vec{B}$

and also operations on vectors $\underbrace{q\vec{E}}_{\text{multiply}} \overset{\text{addition}}{+} \underbrace{\vec{v} \times \vec{B}}_{\text{2 vectors}}$

q: a # $\vec{E}$: a vector $\times$: multiply

We need
(1) adding / multiplying vectors
(2) calculus with vectors

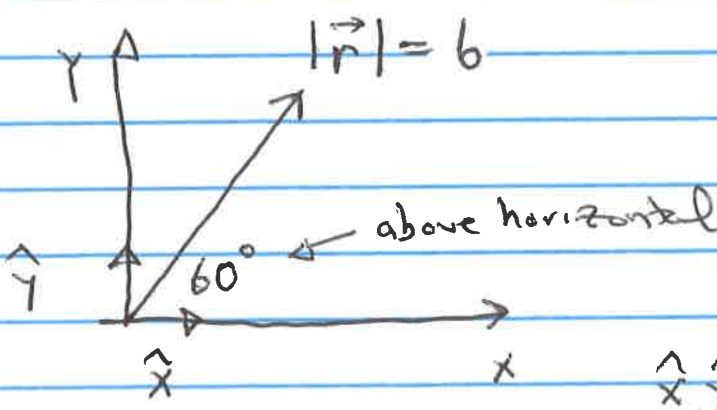
You have seen both before but let's review, especially (2).

Two ways to represent vectors

(1) Magnitude + direction

(2) Components

An object is located 6 meters from origin along line 60° above horizontal



(1)

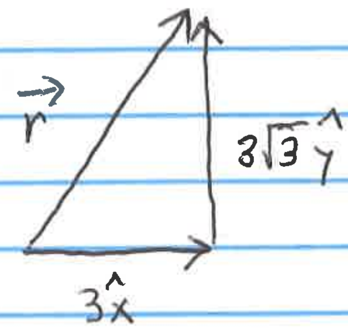
$\hat{x}, \hat{y}$ unit vectors

(2) $\vec{r} = 3\hat{x} + 3\sqrt{3}\,\hat{y}$

V3

With the understanding that

* $3\hat{x}$ ← vector of same direction as $\hat{x}$ but 3 times as long



add vectors by placing "tail" of one at "head" of another.

Q Does order matter?

Q Where did $3\hat{x}$ and $3\sqrt{3}\,\hat{y}$ come from?

$6\cos 60$ $\quad$ $6\sin 60$

$\frac{1}{2}$ $\quad$ $\sqrt{3}/2$

So we have reviewed
* adding vectors
* multiplying vectors by a number.

Another way to add vectors

Q

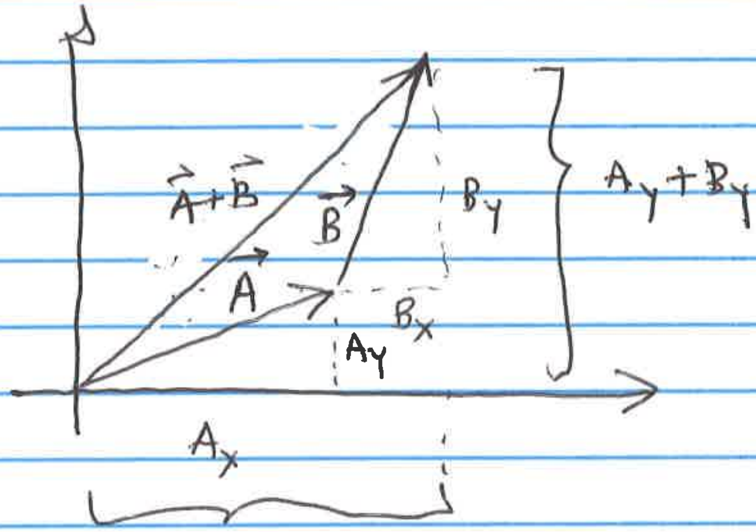
$$\vec{A} = A_x\hat{x} + A_y\hat{y}$$

$$\vec{B} = B_x\hat{x} + B_y\hat{y}$$

$$\vec{A}+\vec{B} = (A_x+B_x)\hat{x} + (A_y+B_y)\hat{y}$$

V4 Equivalent to tail on head definition

From this picture



Q

$\vec{A}+\vec{B}$ vs $\vec{B}+\vec{A}$

$\boxed{\text{V4A}}$

Two ways to multiply two vectors

θ_{AB} = angle between $\vec{A}$ and $\vec{B}$

dot product yields a number

$\rightarrow \quad \vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta_{AB} = A_xB_x + A_yB_y + A_zB_z$

Q

$\vec{A}\times\vec{B} \Rightarrow |\vec{A}||\vec{B}|\sin\theta_{AB} = |\vec{A}\times\vec{B}|$

$\perp$ to plane of $\vec{A}, \vec{B}$ = direction

; "right hand rule"

$(\vec{A}\times\vec{B})_x = A_yB_z - A_zB_y$

Q $\quad \vec{A}\cdot\vec{B}$ vs $\vec{B}\cdot\vec{A}$

$(\vec{A}\times\vec{B})_y =$

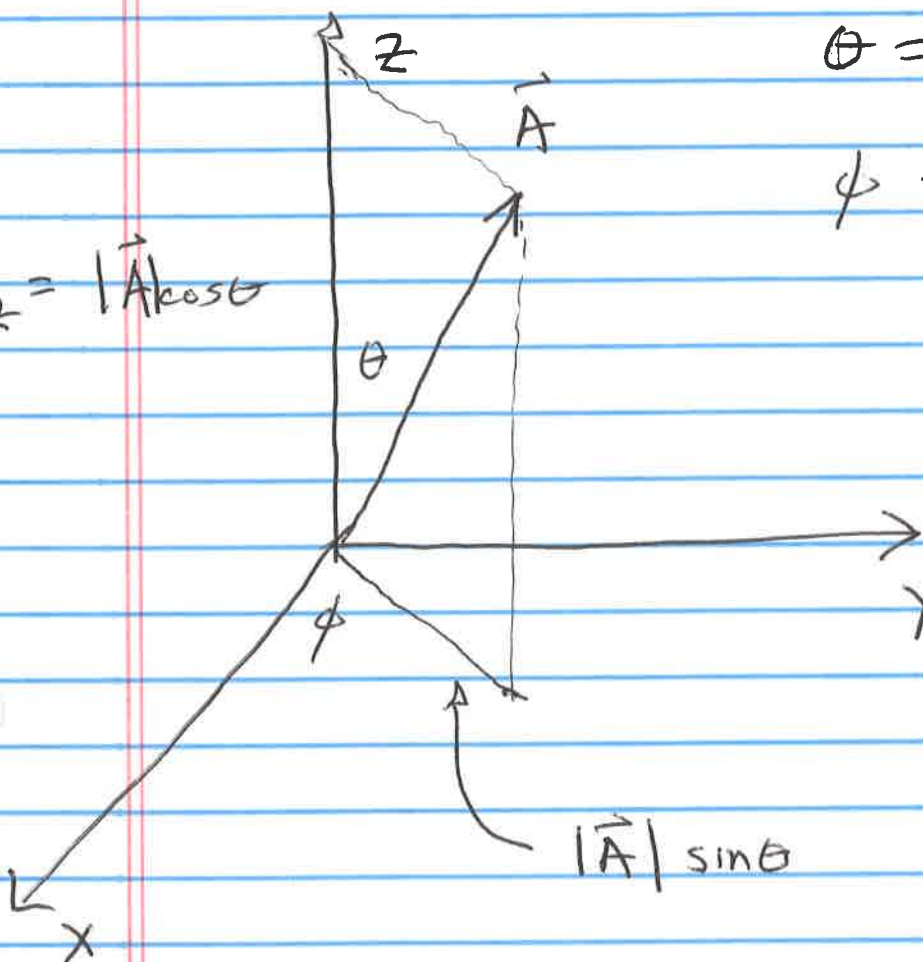
Q $\quad \vec{A}\times\vec{B}$ vs $\vec{B}\times\vec{A}$?

$(\vec{A}\times\vec{B})_z =$

V4A

In 3D

$A_z = |\vec{A}|\cos\theta$



θ = angle wrt $\hat{z}$ axis

ϕ = angle of projection of $\vec{A}$ onto xy plane wrt x axis

$A_x = |\vec{A}|\sin\theta\cos\phi$

$A_y = |\vec{A}|\sin\theta\sin\phi$

$$\underbrace{|\vec{A}|}_{\text{length}} \quad \underbrace{\theta,\phi}_{\text{direction}} \iff \underbrace{A_x \; A_y \; A_z}_{\text{components}}$$

V5

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}\,(A_y B_z - A_z B_y) + \hat{y}\,(A_z B_x - A_x B_z) + \hat{z}\,(A_x B_y - A_y B_x)$$

determinant

$\boxed{\text{V5A}}$

$\vec{A}\cdot(\vec{B}\times\vec{C})$ is a #: the volume of parallelepiped formed by $\vec{A}, \vec{B}, \vec{C}$!

$\hat{x}$ component of

$$A_x(B_y C_x - B_x C_y) + A_y(\quad) + A_z(\quad)$$

$$= \vec{B}\cdot(\vec{C}\times\vec{A})$$

$$= \vec{C}\cdot(\vec{A}\times\vec{B})$$

$\vec{A}\times(\vec{B}\times\vec{C})$ is a vector

$$= \vec{B}\,(\vec{A}\cdot\vec{C}) - \vec{C}\,(\vec{A}\cdot\vec{B})$$

we know $\vec{A}\times(\vec{B}\times\vec{C})$ must be $\perp$ to $\vec{A}$

so it makes sense it has no piece $\parallel$ to $\vec{A}$!

VSA

Something new

use 1,2,3 instead of x y z

$\hat{x}$	A_x	A_1
$\hat{y}$	A_y	A_2
$\hat{z}$	A_z	A_3

"Kronecker" delta $\quad \delta_{ij} = 1 \quad$ if $i=j$

$\qquad\qquad 0 \quad$ if $i \neq j$

$$\delta_{11} = \delta_{22} = \delta_{33} = 1$$

$$\delta_{12} = \delta_{21} = \delta_{13} = \ldots = 0$$

$\epsilon_{ijk} = +1 \quad$ if $i\,j\,k$ all different and $i\,j\,k$ an even permutation of 123

$-1 \quad$ odd permutation

$\phi \quad$ any of i, j, k equal

$\epsilon_{123} = +1 \qquad \epsilon_{213} = -1 \qquad \epsilon_{231} = +1$

$\epsilon_{113} = 0 \qquad \epsilon_{212} = 0 \qquad$ etc ...

$$(A \times B)_i = \sum_{jk}{}' \epsilon_{ijk} A_j B_k = \epsilon_{ijk} A_j B_k$$

"Einstein summation convention"

V5B

$$(\vec{A} \times \vec{B})_x = A_Y B_Z - A_Z B_Y$$

$$(\vec{A} \times \vec{B})_1 = A_2 B_3 - A_3 B_2$$

$$(\vec{A} \times \vec{B})_1 = \epsilon_{1jk} A_j B_k$$ ← get zero unless $j, k = 2, 3$

$\epsilon_{123} = +1$

$\epsilon_{132} = -1$

$$\Longrightarrow A_2 B_3 - A_3 B_2$$

Very useful identity $\quad \epsilon_{ijk} \epsilon_{k\ell n} = \delta_{i\ell} \delta_{jn} - \delta_{in} \delta_{j\ell}$

This formalism allows you to prove vector identities in a very elegant way, and is widely used in advanced (eg graduate level) physics courses!

Eg last eqn on page V5

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \epsilon_{ijk} A_j (\vec{B} \times \vec{C})_k$$

$$= \epsilon_{ijk} A_j \epsilon_{k\ell n} B_\ell C_n$$

$$= (\delta_{i\ell} \delta_{jn} - \delta_{in} \delta_{j\ell}) A_j B_\ell C_n$$

$$= A_n B_i C_n - A_\ell B_\ell C_i = (\vec{A} \cdot \vec{C}) B_i - (\vec{A} \cdot \vec{B}) C_i$$

V6

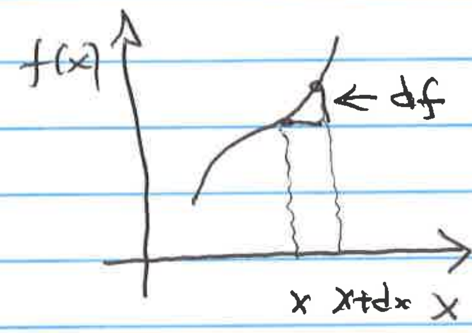
Differentiation + Integration with vectors

slope of f(x)

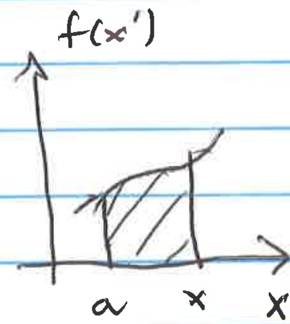
Review $df = \frac{df}{dx}\, dx$

change in function f(x)

if x changes by dx



$\int_a^x f(x')\,dx'$ = area under $f(x')$ from $x'=a$ to $x'=x$



Amazingly derivative and integral are related!

Q $\frac{d}{dx}\int_a^x f(x')\,dx' = f(x)$ How to prove?

put another way $\underbrace{\int_a^b \frac{df}{dx}\,dx}_{\text{integral}} = \underbrace{f(b)-f(a)}_{\text{value of f at boundaries}}$

V7

"Scalar function"

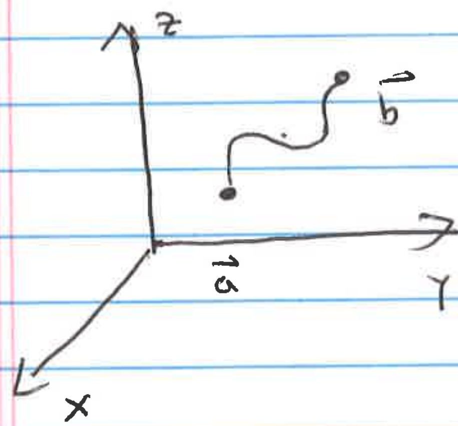
Differentiation of $f(x, y\ z)$

$\boxed{\text{gradient}}$ $$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

Q

$\vec{\nabla} f$ points in direction of maximum increase of f

$$\int_{\vec{a}}^{\vec{b}} \vec{\nabla} f \cdot d\vec{l} = \underbrace{f(\vec{b}) - f(\vec{a})}_{\text{values at boundary}}$$ independent of path taken!



sort of like

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

V7A

What about vector function?

$$\vec{V}(x,y,z) = V_x(x,y,z)\,\hat{x} + V_y(x,y,z)\,\hat{y} + V_z(x\,y\,z)\,\hat{z}$$

Q

$\boxed{\text{divergence}}$ $$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$\boxed{\text{curl}}$ $$\vec{\nabla}\times\vec{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{x} + \ldots = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

V 7A

Check Fundamental Theorem for gradient

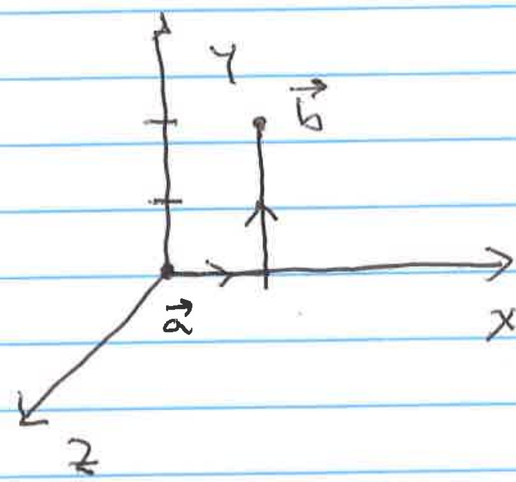
$$F(x,y) = x^2 y^3$$

$\vec{a} = (0,0,0)$ $\vec{b} = (1,2,0)$

Choose a path

part 1 (000) to (100)

part 2 (100) to (120)



$$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$\vec{\nabla} f = \underbrace{2xy^3}_{\partial f/\partial x}\,\hat{x} + \underbrace{3x^2y^2}_{\partial f/\partial y}\,\hat{y}$$

part 1 : $\vec{dl} = dx\,\hat{x}$

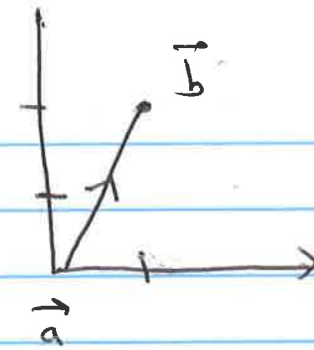
part 2 : $\vec{dl} = dy\,\hat{y}$

$$\int_a^b \vec{\nabla} f \cdot \vec{dl} = \int_0^1 2x\underbrace{y^3}_{y=0} dx + \int_0^2 3\underbrace{x^2}_{x=1} y^2 dy = y^3\Big|_0^2 = 8$$

$f(b) = 8$ $f(a) = 0$ ✓✓

V7B

Another path is $y = 2x$.



$$\vec{dl} = dx\,\hat{x} + dy\,\hat{y}$$

where $dy = 2dx$

$y = 2x$

$$\int_0^1 2x\underbrace{y^3}_{2x}\,dx + 3x^2\underbrace{y^2}_{2x}\,\underbrace{dy}_{2dx}$$

$$= \int_0^1 16x^4\,dx + 24x^4\,dx = \int_0^1 40x^4\,dx = 8x^5\Big|_0^1$$

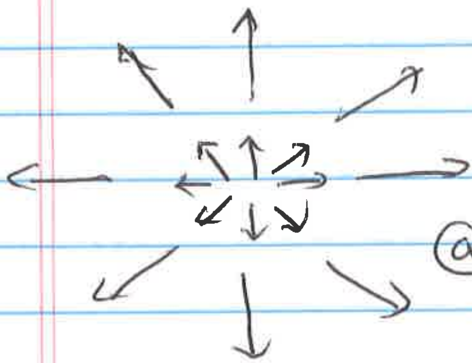
$= 8$ ✓

V8

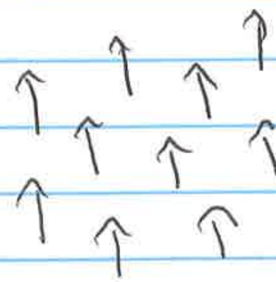
(Q) Geometrical interpretation of divergence and curl?

drop dust in pond. If it spreads out, positive divergence, if it contracts, negative divergence

(Q)

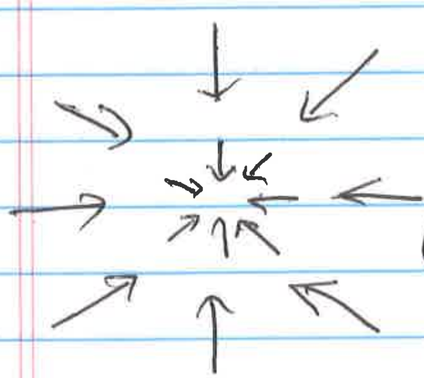


(a)



(b)

(c)



(d)

$\boxed{\text{WHY}}$ (Q)

a) $\vec{v} = x\hat{x} + y\hat{y} + z\hat{z}$

$\vec{\nabla}\cdot\vec{v} = 3$

b) $\vec{v} = 3\hat{y}$ $\qquad \vec{\nabla}\cdot\vec{v} = 0$

c) $\vec{v} = y\hat{y}$ $\qquad \vec{\nabla}\cdot\vec{v} = 1$

d) $\vec{v} = -x\hat{x} - y\hat{y} - z\hat{z}$ $\qquad \vec{\nabla}\cdot\vec{v} = -3$

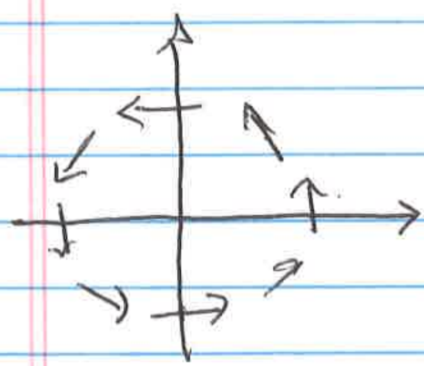
V9

aka "rotation"

curl measures "swirling" of vectors

imagine putting toothpick in vector field.

Would it spin?



$\vec{v} = -y\hat{x} + x\hat{y}$

$v_x = -y$
$v_y = +x$
$v_z = 0$

$\vec{\nabla}\cdot\vec{v} = 0$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & +x & 0 \end{vmatrix} = 2\hat{z}$$

"counterclockwise is positive"

pictures on page 8

a) $$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0$$ no "swirling"

b) $$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 3 & 0 \end{vmatrix} = 0$$ no 'swirling'