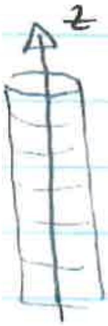
B-0

Bessel Function (Laplacian in cylindrical Coordinates)

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}$$



so long cylinder · rotational symmetry is axis

We will first consider case of no z, ϕ dependence since Bessel functions already appear there. Then we will consider less symmetrical case

{ Schroedinger Eqn in "pillbox" ← class
{ Laplace Eqn in "pillbox" ← HW

$$D\nabla^2\psi = \partial\psi/\partial t \qquad \psi = A(\rho)\,e^{-\alpha t}$$

$$D\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)A\cancel{e^{-\alpha t}} = -\alpha A\cancel{e^{-\alpha t}}$$

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\alpha}{D}\right)A(\rho) = 0$$

$$\left[\rho^2\frac{\partial}{\partial\rho^2} + \rho\frac{\partial}{\partial\rho} + \left(\frac{\alpha}{D}\right)\rho^2\right]A(\rho) = 0$$

"k^2"

B-1

$\boxed{\text{Bessel Functions}}$ ← soln of $\nabla^2 \psi = -k^2 \psi$ also stat mech!

Thus In solving diffusion Eqn for cylindrical geometry

we encounter

$$\left[x^2 \frac{d}{dx^2} + x\frac{d}{dx} + k^2 x^2\right] u(x) = 0$$

The solns are a special case $(n=0)$ of the Bessel Eqn

$$* \qquad \left[x^2 \frac{d}{dx^2} + x\frac{d}{dx} + (x^2 - n^2)\right] J_n(x) = 0$$

Ie $\quad u(x) = J_0(kx).$

{ note that if $s \equiv kx$ $\quad s\frac{d}{ds} = x\frac{d}{dx}$ and $s^2\frac{d^2}{ds^2} = x^2\frac{d^2}{dx^2}$

So that $\left(x^2 d^2/dx^2 + x\, d/dx + k^2x^2\right) \Xi(kx)$

$$= \left(s^2\frac{d^2}{ds^2} + s\, d/ds + s^2\right) u(s)$$

↖ Bessel for $n=0$ }

What are Bessel functions, the solns to $*$?!

Basically they are like sines and cosines (oscillating)

except they also decay, and zeroes not evenly spaced

B-2

Sines and cosines can be defined in different ways.

Most simply $\quad d^2/dx^2\, u(x) = -k^2 u$

But also

$\frac{d^2u}{dx^2} = +k^2 u$

$u = \sinh, \cosh$

series?

$$\left\{ \begin{aligned} \sin x &= x - x^3/3! + x^5/5! - \dots \\ \cos x &= 1 - x^2/2! + x^4/4! - \frac{x^6}{6!} \end{aligned} \right.$$

- $\sin, \cos$ complete $\Rightarrow$ Fourier rep
- $\frac{1}{\pi}\int_0^{2\pi} \sin nx \sin mx\, dx = \delta_{nm}$
- $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-k')x}\, dx = \delta(k-k')$

For Bessel functions

$$J_n(x) = \sum_s 1/s!\,(s+n)!\,(-1)^s \left(\frac{x}{2}\right)^{2s+n}$$

eg $\quad J_0(x) = \sum_s 1/(s!)^2 (-1)^s \left(\frac{x}{2}\right)^{2s}$

$$= 1 - \frac{x^2}{4} + \frac{x^4}{4\cdot 16} - \frac{x^6}{36\cdot 64}$$

$$J_1(x) = \sum_s 1/s!\,(s+1)!\,(-1)^s \left(\frac{x}{2}\right)^{2s+1}$$

$$= \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{96} -$$

Where does this series expansion come from?

$$\frac{d}{dx}\sin x = 1 - x^2/2! + x^4/4!$$

$$d^2/dx^2 \sin x = -x + x^3/3! = -\sin x$$

$$\frac{d}{dx} J_0(x) = -\frac{x}{2} + \frac{x^3}{16} - \frac{x^5}{6\cdot 64}$$

$$\frac{d^2}{dx^2} J_0(x) = -\frac{1}{2} + \frac{3}{16}x^2 - \frac{5x^4}{6\cdot 64}$$

$$x^2 \frac{d^2}{dx^2} J_0(x) + x \frac{d}{dx} J_0(x) + x^2 J_0(x)$$

$$= -\frac{1}{2}x^2 + \frac{3}{16}x^4 - \frac{5}{6\cdot 64}x^6$$

$$-\frac{x^2}{2} + \frac{x^4}{16} - \frac{x^6}{6\cdot 64}$$

$$+ x^2 - \frac{x^4}{4} + \frac{x^6}{4\cdot 16} = 0$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$0 \quad 0 \quad 0$$

So (1) Bessel DE
(2) SERIES
(3) other...

~~So it works but where does it come from~~

$$e^{x/2(t - 1/t)} = \sum_n J_n(x) t^n$$

generating function for Bessel

Can show (1) gives series of page B-2

(2) Satisfies Bessel DE of page B-1

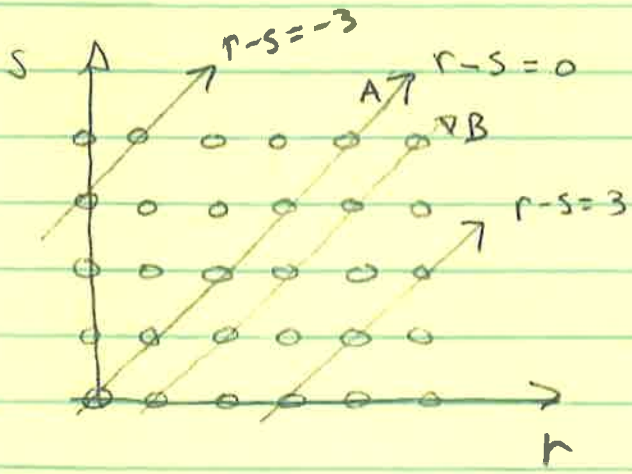
Basically a bunch of atedious algebra...

~~HW: Use g(x,t)~~ to prove $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$

B-4

Proof of (1)

$$e^{xt/2}\, e^{-x/2t} = \sum_{r=0}^{\infty} \left(\frac{x}{2}\right)^r \frac{t^r}{r!} \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^s (-1)^s \frac{t^{-s}}{s!}$$



t^{r-s}

$n = r - s$

$s = r - n$

$r = s + n$

can relabel double sum using n and s instead

n can take any value $-\infty$ to $+\infty$

If $n \geq 0$ s will start at ϕ

$$\sum_{n=-\infty}\sum_{s=-n}^{\infty} + \Big) \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{s+n} \frac{t^{s+n}}{(s+n)!} \left(\frac{x}{2}\right)^s (-1)^s \frac{t^{-s}}{s!}$$

$n=0 \quad s=0,1,2,3\ldots \rightarrow$ line A since $r = s+n = s$

$n=1 \quad s=0,1,2,3\ldots \rightarrow$ ~~line B same $r = s+n = s+1$~~

Focus on this to get J_n for $n>0$

$$= \sum_{n=0}^{\infty} t^n \sum_{s=0}^{\infty} \frac{1}{s!}\frac{1}{(s+n)!}(-1)^s \left(\frac{x}{2}\right)^{2s+n}$$

Considering $n<0$ in same way leads to

$$J_n(x) = -J_{-n}(x)$$

B-5

$$g(x,t) = e^{x/2(t-1/t)}$$

proof of (2)

$$d/dx\ g(x,t) = 1/2(t-1/t)\, e^{x/2(t-1/t)}$$

$$d^2/dx^2\ g(x,t) = 1/4(t-1/t)^2 e^{x/2(t-1/t)}$$

$$\therefore \left(x^2 \frac{d^2}{dx^2} + x\frac{d}{dx} + x^2 - n^2\right) g(x,t)$$

$$= \left[x^2 \frac{1}{4}\left(t-\frac{1}{t}\right)^2 + x\,\frac{1}{2}\left(t-\frac{1}{t}\right) + x^2 - n^2\right] e^{x/2(t-1/t)}$$

$$= \left[x^2 \frac{1}{4}t^2 - \frac{x^2}{2} + \frac{x^2}{4t^2} + \frac{xt}{2} - \frac{x}{2t} + x^2 - n^2\right] \quad ''$$

...

Further connections to familiar $\sin x$ and $\cos x$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \cdots \qquad \cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} \cdots$$

$$= \sum_s \frac{1}{(2s)!} x^{2s} (-1)^s \qquad = \sum_s \frac{1}{(2s)!} x^{2s}$$

$$\cos(ix) = \cosh x$$

similarly

$$J_n(x) = \sum \frac{1}{s!} \frac{1}{(s+n)!} \left(\frac{x}{2}\right)^{2s+n} (-1)^s$$

$\downarrow$ eliminated

$$\rightarrow I_n(x) =$$

"Modified Bessel Function"

others Neumann, Hankel are essentially like

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$\cos x, \sin x$ vs $e^{ix}; e^{-ix}$

B-7

An interesting identity comes from replacing t in generating function by $e^{i\theta}$

$$\frac{x}{2}\left(t - 1/t\right) \to \cancel{x}\cos\theta$$

$$e^{x\cos\theta} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\theta}$$

We will see next quarter that this allows us to solve a very interesting statistical mechanics problem, "the XY chain" in terms of Bessel functions.

Completeness + Orthogonality of Bessel Functions

$$\left[x^2 \frac{d^2}{dx^2} + x\frac{d}{dx} + (k^2x^2 - n^2)\right] J_n(kx) = 0$$

Recall Sturm-Liouville theory

$$\mathcal{L} \equiv p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$$

is Hermitian if $p_1(x) = p_0'(x)$

In such a situation the eigenfunctions $f_\lambda(x)$

$$\mathcal{L} f_\lambda(x) = \lambda f_\lambda(x)$$

are a complete set:

$$g(x) = \int d\lambda \; c(\lambda) f_\lambda(x)$$ ← any g can be expanded in f_λ

and f_λ are also orthogonal:

$$\int dx\, f_\lambda(x) f_{\lambda'}(x) = \delta_{\lambda,\lambda'} \text{ or } \delta(\lambda-\lambda')$$

discrete λ — continuous λ.

The Bessel differential operator is Hermitian only after dividing by the "weight function" $w(x) = x$

$$x \frac{d}{dx} + \frac{d}{dx} + (k^2 x^2 - n^2/x) \qquad J_n(kx) = 0$$

$$\uparrow \qquad\qquad \uparrow$$

$$p_0(x) = x \qquad p_1(x) = 1 \qquad p_1 = p_0'$$

In this case, as we discussed, the eigenfunctions obey a generalized orthogonality

$$\int w(x) f_\lambda(x) f_{\lambda'}(x)\, dx = \delta_{\lambda\lambda'} \qquad \delta(\lambda - \lambda')$$

Let's work out how this occurs explicitly for Bessel functions. Consider problem where we require functions to vanish at $\rho = a$ $(x \to \rho)$. When we solved Schroedinger eqn we saw this quantized the K values

$$J_n(\alpha_{nm}\, \rho/a) \qquad \text{where } \alpha_{nm} \text{ are mth roots of } J_n\text{:}$$

$$J_n(\alpha_{nm}) = 0$$

Aside: Orthogonality and completeness depends not only on $\mathcal{L}$ but also on boundary conditions $\frac{d^2}{dx^2} f = -k^2 f \quad 0 < x < a \to \sin \frac{n\pi x}{a}$

$$f(x) = 0$$

$$x = 0, a$$

8B-3

The derivation below follows very closely last quarter's derivation of the $\rho_1 = \rho_0'$ Hermiticity condition

$\cdot J_n(\alpha_{nm'}\rho/a)$

$$\left[\rho \frac{\partial^2}{\partial \rho^2} J_n\left(\alpha_{nm}\frac{\rho}{a}\right) + \frac{\partial}{\partial \rho} J_n\left(\alpha_{nm}\frac{\rho}{a}\right) + \frac{\alpha_{nm}^2 \rho}{a^2} - \frac{\nu^2}{\rho}\right] J_n\left(\alpha_{nm}\frac{\rho}{a}\right) = 0$$

$$\left[\rho \frac{\partial^2}{\partial \rho^2} J\left(\alpha_{nm'}\frac{\rho}{a}\right) + \frac{\partial}{\partial \rho} J_n\left(\alpha_{nm'}\frac{\rho}{a}\right) + \frac{\alpha_{nm'}^2}{a^2}\rho - \frac{\nu^2}{\rho}\right] J_n\left(\alpha_{nm'}\frac{\rho}{a}\right) = 0$$

$\cdot J_n(\alpha_{nm}\rho/a)$

and subtract

$$J_n\left(\alpha_{nm'}\frac{\rho}{a}\right) \frac{\partial}{\partial \rho}\left[\rho \frac{\partial}{\partial \rho} J_n\left(\alpha_{nm}\frac{\rho}{a}\right)\right]$$
$$- J_{nm}(\alpha_{nm}\rho/a) \frac{\partial}{\partial \rho}\left[\rho \,\partial/\partial\rho\, J_n(\alpha_{nm'}\rho/a)\right] = \frac{\alpha_{nm}^2 - \alpha_{nm'}^2}{a^2}\rho J_n\left(\alpha_{nm}\frac{\rho}{a}\right) J_n\left(\alpha_{nm'}\frac{\rho}{a}\right)$$

Integrate from 0 to a and then integrate by parts

The "integral term" vanishes because one has two identical pieces differing by a $\ominus$ sign. The "surface term" vanishes at $\rho = \phi$ because of the ρ factor and at $\rho = a$ because of $\alpha_{nm}\rho/a$ argument.

Thus as long as $\alpha_{nm}^2 \neq \alpha_{nm'}^2$ we have

$$\int_0^a \rho J_n(\alpha_{nm}\rho/a) J_n(\alpha_{nm'}\rho/a)\, d\rho = 0$$

ORTHOGONALITY

Normalization $$\int_0^a \left[J_n(\alpha_{nm}\rho/a)\right]^2 \rho\, d\rho = \frac{a^2}{2}\left[J_{n+1}(\alpha_{nm})\right]^2$$

$\boxed{\text{Exercise}}$ from recurrence reln (HW 1-1)

Summary we can expand $f(\rho)$ $0 < \rho < a$ $\quad f(0) = 0$, $f(a) = 0$

$$f(\rho) = \sum_{m=1}^{\infty} C_{nm} J_n(\alpha_{nm} \rho/a)$$

$$C_{nm} = \frac{2}{a^2 (J_{n+1}(\alpha_{nm}))^2} \int_0^a f(\rho) J_n(\alpha_{nm} \rho/a) \rho \, d\rho$$

Q: why are $\{J_n(\alpha_{nm} \rho/a)\}$ complete for each n?

Shouldn't we have to include all $\underline{\underline{n}}$ in expansion?

A: Different δ, PDE for each n

DE-4

Do SQUARE $\square$ first

No 11.1.28

$\boxed{\text{Diffusion Eqn - Critical Mass}}$

We wrote down diffusion eqn in absence of any sources

$$D \nabla^2 \psi = \frac{\partial \psi}{\partial t} \quad \Big\} \text{ Fourier soln}$$

Stuff (ink, temperature, ...) just spreads out.

Like free particle Sch Eqn

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = \frac{\hbar}{i}\frac{\partial \psi}{\partial t} \quad \Big\} \text{ free propagator, path integral}$$

Except for "i". Now add a source term

$$D \nabla^2 \psi + \lambda \psi = \frac{\partial \psi}{\partial t}$$

If $\psi(r,t) = \psi(t)$ (no spatial dependence)

$$\psi(t) \sim e^{\lambda t} \qquad \text{exponential growth/decay}$$

[What about Sch. case: What does λ term correspond to? Does it affect physics?]

HEAT

SEE DE-4

Problem: ~~Melting~~ Bar with internal heat source
have a given amount of material. What
cross section most likely to melt $\square$ vs $\bigcirc$?

At end: More dramatic example: neutron diffusion in uranium rod

$\swarrow e^{-\alpha t}$

assume $\quad \psi(\vec{r},t) = u(\vec{r})\, f(t)$

$$D\nabla^2 u + (\lambda+\alpha)u = 0$$

Infinite cylinder $\quad u(\vec{r}) = u(\rho, z, \phi) = u(\rho)$

Cylindrical coordinates

$h_1 =$	ρ	ϕ
$h_2 =$	1	ρ
$h_3 =$	1	z

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\frac{h_2 h_3}{h_1}\frac{\partial}{\partial q_1} + \cdots\right]$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \frac{\partial}{\partial z}\rho\frac{\partial}{\partial z} + \frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\right]$$

no ϕ, z dependence $\Rightarrow \quad \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}$

$$\left[D\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right) + \underbrace{(\lambda+\alpha)}_{\parallel\; Dk^2}\right]u = 0$$

$\lambda + \alpha = Dk^2$

$\alpha = Dk^2 - \lambda$

$$\left[\rho^2\frac{\partial^2}{\partial\rho^2} + \rho\frac{\partial}{\partial\rho} + k^2\rho^2\right]u = 0$$

$$\left[x^2\frac{d^2}{dx^2} + x\frac{d}{dx} + x^2 - n^2\right]y = 0 \qquad y = J_n(x)$$

$\therefore \quad u = J_0(k\rho)$

analog of e^{-ikx} in $d=1$ or $\cos k_n x \cos k_m y$ on next page

$\swarrow$

$$u(\rho,\phi,z,t) = \int dk\, a(k)\, J_0(k\rho)\, e^{-(Dk^2-\lambda)t}$$

Example:

$\left(x\frac{d}{dx} + x\right)y = 0$

~~$(d/dx + 1)y = 0$~~ skip

~~$dy/dx = -y$~~

$y = e^{-x}$

$\left(x\frac{d}{dx} + kx\right)y = 0$

$y = e^{-kx}$

$dy/dx = -ke^{-kx}$

Yes! $\int_0^\infty J_\nu(\alpha\rho)\, J_\nu(\alpha'\rho) = \frac{1}{\alpha}\delta(\alpha-\alpha')$

What question naturally arises?

Are $J_n(k\rho)$ complete? Is Bessel Hermitian

temperature
neutron density = 0 at $\rho = a$ — diameter of cylinder

Ka = roots of J_0

not all be allowed! $\left\{ \quad u(\rho,\phi,z,t) = \sum_n a_n J_0(k_n\rho)\, e^{-(Dk_n^2-\lambda)t} \right.$

→ k_n obeying $J_0(k_n a) = 0$

Exponential growth occurs when

$\lambda > Dk_n^2$

$Dk_n^2 - \lambda < 0$

again λ large is bad or D small is bad.

$k_n^2 < \lambda/D$

$(2.405)^2/a^2 < \lambda/D$

$a^2 > (2.405)^2 D/\lambda$

$k_n a = 2.405$
$= 5.520$
$= 8.654$

Exponential growth occurs first at smallest root

$a_{critical} = 2.405\sqrt{\frac{D}{\lambda}}$

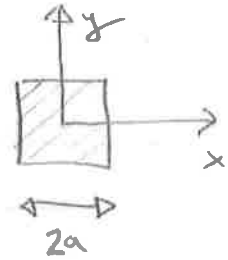
depends on diffusion rate of neutrons (a increases)
and λ rate of production of neutrons (a decreases)

Interesting points ① What happens at $t = \infty$ only $T = 0$ or $T = \infty$
no nontrivial steady state?! Answer
Boundary conditions maybe should be $\partial T/\partial x$ not T

② Can A_{00} every be zero?
No. See ~~next~~ rectangle example especially
Initial Temp ≥ 0 at all points so A_{00}

so melting is indep of initial conditions. All IC have $A_{00} \neq 0$

DE-4



Square cross section

$$D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\lambda\right)\psi=\frac{\partial\psi}{\partial t}$$

$$\psi = u(r)\,e^{-\alpha t}$$

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{\lambda+\alpha}{D}\right)u(x,y)=0$$

$k_n a = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$$u(x,y) = \cos k_n x \cos k_m y$$

$\cos(k_n a) = 0$

$k_n = \frac{\pi, 3\pi, 5\pi, \ldots}{2a}$

$= \frac{\pi}{2a}(1, 3, 5, \ldots$

$$-k_n^2 - k_m^2 + \frac{\lambda+\alpha}{D} = 0$$

$$\alpha = -\lambda + D(k_n^2 + k_m^2)$$

$$= -\lambda + D\frac{\pi^2}{4a^2}\left[(2n+1)^2+(2m+1)^2\right]$$

$= -\lambda + \frac{D\pi^2}{4a^2}(1+1)$

$-\lambda + D\frac{\pi}{4a^2}(1+9)$

$\vdots$

if λ small okay

if "a" small we're okay. If D is large this is large + and $\alpha > 0$: decay

Bad news when $\alpha < 0$. Easiest to occur at $n = m = 0$

$$\lambda = D\frac{\pi^2}{2a^2}$$

$$a^2 = \frac{D\pi^2}{2\lambda} \qquad a = \frac{\pi}{\sqrt{2}}\sqrt{\frac{D}{\lambda}}$$

$A^{circle}_{crit} = \pi a^2 = (2.405)^2 \pi\, D/\lambda = 18.2\, D/\lambda$

$A^{rect}_{crit} = (2a)^2 = 4\frac{\pi^2}{2}\frac{D}{\lambda} = 19.7\, D/\lambda$

radiators 

DE-5.

$$\int dk \sum_{\ell m} Y_{\ell m}(\phi,\theta) \underbrace{\frac{J_{\ell+1/2}(kr)}{\sqrt{r}}}_{r^\ell,\ r^{-(\ell+1)}} A_{\ell m k}$$

Sphere of Uranium

$$\left[\nabla^2 + \frac{(\lambda+\alpha)}{D}\right] u(r,\theta,\phi) = 0$$

arbitrary coefficients

$$\psi(r,\theta,\phi,t) = \sum_{\ell,n} A_{\ell n} P_\ell(\cos\theta) \frac{1}{\sqrt{r}} J_{\ell+\frac{1}{2}}(k_{\underbrace{\ell n}} r) e^{-(Dk_{\ell n}^2 - \lambda)t}$$

two indices

$$k^2 = \frac{\lambda+\alpha}{D}$$

no ϕ dependence $Y_\ell^m(\theta,\phi) \to P_\ell(\cos\theta)$

Legendre polynomial

$$\alpha = Dk^2 - \lambda$$

$$J_{\ell+\frac{1}{2}}[k_{\ell n}\, a] = 0$$

radial eqn ~ Bessel eqn with $n = \ell + 1/2$

with $1/\sqrt{r}$ in front

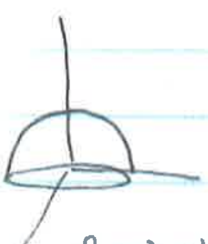
Smallest $k_{\ell n}$ is for $\cancel{J_{1/2}}$ $J_{1/2}$ 3.1416 (Abramowitz + Stegun 46.7)

← back of preceding

$$\left(D\frac{\pi^2}{a^2} - \lambda\right) < 0 \qquad \frac{D\pi^2}{a^2} < \lambda \qquad \frac{a^2}{\pi^2 D^2} > \frac{1}{\lambda}$$

$$a > \pi\sqrt{\frac{D}{\lambda}}$$

$$V = \frac{4}{3}\pi a^3 = \frac{4}{3}\pi\ \pi^3\left(\frac{D}{\lambda}\right)^{3/2} = \frac{4\pi^4}{3}\left(\frac{D}{\lambda}\right)^{3/2}$$



Hemisphere $\ell + \frac{1}{2} = 3/2$ 4.493

only odd ℓ allowed

$P_\ell(\cos \pi/2) = 0$

$P_\ell(0) = 0$

$$\left.\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \tfrac{3}{2}x^2 - \tfrac{1}{2} \end{aligned}\right\}$$

$$a > 4.493\sqrt{\frac{D}{\lambda}}$$

bigger as expected!

$$\left(\frac{4.493}{3.142}\right)^3 = 2.925$$

REVIEW STAT MECH

Bessel Function in Statistical Mechanics!

1-d chain of magnetic moments which can rotate in a plane

$$E = -K \sum_{\ell=1}^{N} \cos(\theta_\ell - \theta_{\ell+1})$$

ground state is $\theta_\ell = $ const $\quad E = -NJ$

"ferromagnetism"

$$Z = \sum_{\text{states}} e^{-\beta E}$$

$$= \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \ldots \int_0^{2\pi} d\theta_N \; e^{+\beta K \sum_{\ell=1}^{N} \cos(\theta_\ell - \theta_{\ell+1})}$$

$$= \quad '' \quad e^{\beta K \cos(\theta_1 - \theta_2)} e^{\beta K \cos(\theta_2 - \theta_3)} \ldots$$

$$\int_0^{2\pi} d\theta_2 \, e^{\beta K \cos(\theta_1 - \theta_2)} e^{\beta K \cos(\theta_2 - \theta_3)}$$ reminds us of which

$$M^2(\theta_1, \theta_3) = \int d\theta_2 \, M(\theta_1, \theta_2) \, M(\theta_2, \theta_3)$$

$$Z = \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_N \, M^{N-1}(\theta_1, \theta_N)$$

PBC $\quad K\cos(\theta_1 - \theta_N)$ term

In finite dim matrices + Hilbert space Already encountered

$\langle x | p \rangle = e^{ipx}$

$\langle x | \psi \rangle = \psi(x)$

$\langle x | p | \psi \rangle = \frac{\hbar}{i} \frac{\partial \psi}{\partial x}$

$\langle x | p | x' \rangle \langle x' | \psi \rangle$

$\delta(x-x') \frac{\hbar}{i} \partial/\partial x'$

SM-2A

Hw:

$$e^{\frac{x}{2}(t-1/t)} = \sum J_n(x)\, t^n \qquad e^{ix\sin\theta} = \sum J_n(x) e^{i}$$

$$t = e^{i\theta}$$

$$e^{\frac{x}{2}2\cos\theta} = \sum I_n(x)\, e^{in\theta}$$

$$e^{x\cos\theta} = \sum I_n(x)\, e^{in\theta}$$

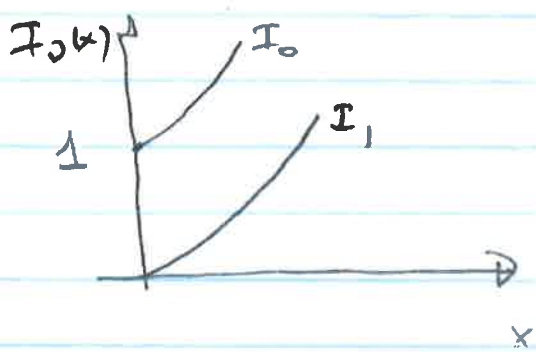
$J_n(x)$ enter $\nabla^2\psi = -k^2\psi$

$$e^{x/2(t+1/t)} = \sum I_n(x)\, t^n$$

$I_n(x)$ enter $\nabla^2\psi = +k^2\psi$

$$J_\nu(x) = \sum_s \frac{1}{s!(s+\nu)!}(-1)^s\left(\frac{x}{2}\right)^{2s+\nu}$$

$$I_\nu(x) = \sum_s \frac{1}{s!(s+\nu)!}\left(\frac{x}{2}\right)^{2s+\nu}$$



$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots$$

anyhow $I_\nu(x) = J_\nu(ix)$ $\qquad \cosh x = \cos(ix)$

Trig analogues $\qquad e^{ix} = \cos x + i\sin x \qquad \sinh x \quad \cosh$

exponential $\qquad$ sines/cosines

$I_\nu \qquad J_\nu$

$$Z = \int_0^{2\pi} d\theta_1 \quad M^N(\theta_1, \theta_1)$$

??

Trace M^N

Eigenvalues of $M \equiv A$

$$Z = \sum_\lambda \lambda^N = \lambda_{max}^N \sum_\lambda \left(\frac{\lambda}{\lambda_{max}}\right)^N$$

$$\to \lambda_{max}^N$$

Eigenvalues?

$$\int_0^{2\pi} M(\theta, \theta_1) f(\theta_1) d\theta_1 = \lambda f(\theta)$$

see pag SM-2A!

$$e^{\beta K \cos(\theta - \theta_1)} = \sum_{n=-\infty}^{\infty} I_n(\beta k) e^{in(\theta-\theta_1)}$$

$$\sum_{n=-\infty}^{\infty} I_n(\beta k) \int_0^{2\pi} e^{in(\theta-\theta_1)} f(\theta_1) d\theta_1 = \lambda f(\theta)$$

$$\sum_{n=-\infty}^{\infty} I_n(\beta k) e^{in\theta} \int_0^{2\pi} e^{-in\theta_1} f(\theta_1) d\theta_1 = \lambda f(\theta)$$

$$f(\theta) = ??$$

5

$f(\theta) = e^{im\theta}$!!

$\int_0^{2\pi} e^{-in\theta_1} e^{im\theta_1} d\theta_1 = 2\pi \delta_{n,m}$

$I_n(\beta K) 2\pi e^{im\theta} = \lambda e^{im\theta}$

$\lambda = 2\pi I_n(\beta K)$

$I_n(x) = \sum \frac{1}{s!(s+n)!} \left(\frac{x}{2}\right)^{2s+n}$

increasing n decreases $I_n(x)$

$\therefore I_0(x) > I_1(x) > I_2(x) > \ldots$

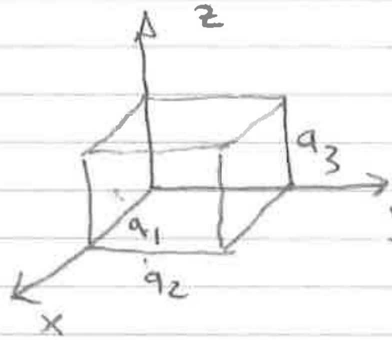
$\lambda_{max} = 2\pi I_0(\beta K)$

$\cancel{J_{1/2}(x) = \sum_s \frac{1}{s!(s+\frac{1}{2})!} (-1)^s \left(\frac{x}{2}\right)^{2s+1/2}}$

Bessel Function Application to Schroedinger Eqn

PROBLEM: Find Energy levels and wave functions of QM particle confined to cylindrical "pillbox".

Lightning review of Rectangular box:



since $V=0$ inside

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi = E\psi$$

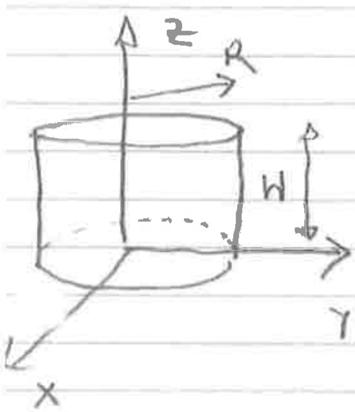
$$\psi(x,y,z) = A\sin k_1 x \,\sin k_2 y \,\sin k_3 z$$

← Automatically ensures $\psi=0$ when $x=0$, $y=0$ or $z=0$

$k_i a_i = n_i \pi$ ← to ensure $\psi=0$ on $x=a_1$, $y=a_2$, $z=a_3$ faces

So $$E = \frac{\hbar^2}{2m}\pi^2\left(\frac{n_1^2}{a_1^2}+\frac{n_2^2}{a_2^2}+\frac{n_3^2}{a_3^2}\right)$$ n_1, n_2, n_3 integers

For cylindrical geometry:



$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}+\frac{\partial^2}{\partial z^2}\right)\psi(\rho,z,\phi) = E\psi$$

$$\psi(\rho,z,\phi) = A(\rho)\sin\frac{l\pi z}{H}\, e^{in\phi}$$

as before $k = \frac{l\pi}{H}$

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{1}{\rho^2}n^2-\frac{l^2\pi^2}{H^2}\right]A(\rho)\cancel{e^{in\phi}}\,\cancel{\sin\frac{l\pi z}{H}} = E A(\rho)\cancel{e^{in\phi}}\,\cancel{\sin\frac{l\pi z}{H}}$$

Thus $\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{1}{\rho^2}n^2-\frac{l^2\pi^2}{H^2}+\frac{2mE}{\hbar^2}\right]A(\rho)=0$

$$\left[\rho^2\frac{\partial^2}{\partial\rho^2}+\rho\frac{\partial}{\partial\rho}+(k^2\rho^2-n^2)\right]A(\rho)=0$$

$$k^2 \equiv \frac{2mE}{\hbar^2}-\frac{l^2\pi^2}{H^2} \qquad \therefore A(\rho) = J_n(k\rho)$$

we need $J_n(kR)=0$

so kR must be a zero of J_n

This quantizes the values of k : $k = \underbrace{k_{nj}}_{j\text{th zero of } J_n}$

$$\boxed{\begin{aligned} E_{njl} &= \frac{\hbar^2}{2m}\left[k_{nm}^2+\frac{l^2\pi^2}{H^2}\right] \\ \psi_{njl}(\rho,\phi,z) &= J_n(k_{nj}\rho)\sin\frac{l\pi z}{H}\,e^{in\phi} \end{aligned}}$$

BS-3

Zeroes of Bessel functions

	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$
1	2.4048	3.8317	5.1356	6.3802
2	5.5201	7.0156	8.4172	9.7610
3	8.6537	10.1735	11.6198	13.0152
4	11.7915	13.3237	14.7960	16.2235