# PHYSICS 110A, WINTER 2018
# ELECTRICITY AND MAGNETISM

## First Midterm Exam

[1.] Let $\mathbf{A} = 2\,\hat{x} - \hat{y}$, $\mathbf{B} = 2\,\hat{x} + \hat{z}$, and $\mathbf{C} = \hat{y} + 3\,\hat{z}$. Evaluate $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ and $\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$, and show they are equal.

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & 0 & 1 \\ 0 & 1 & 3 \end{vmatrix} = \hat{x}(0-1) + \hat{y}(0-6) + \hat{z}(2-0) = -\hat{x} - 6\hat{y} + 2\hat{z}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & -1 & 0 \\ -1 & -6 & 2 \end{vmatrix} = \hat{x}(-2-0) + \hat{y}(0-4) + \hat{z}(-12-1)$$
$$= -2\hat{x} - 4\hat{y} - 13\hat{z}$$

$\vec{A}\cdot\vec{C} = -1$

$\vec{A}\cdot\vec{B} = 4$

$$\left.\begin{aligned} \vec{B}(\vec{A}\cdot\vec{C}) &= (2\hat{x} + \hat{z})(-1) = -2\hat{x} - \hat{z} \\ \vec{C}(\vec{A}\cdot\vec{B}) &= (\hat{y} + 3\hat{z})(4) = 4\hat{y} + 12\hat{z} \end{aligned}\right\} \quad \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) = -2\hat{x} - 4\hat{y} - 13\hat{z}$$

✓

[2.] Given *any* scalar field $f$, show that $\nabla \times \nabla f = 0$. What is the importance of this identity if you are given the vector field $\mathbf{v}$ and are asked to compute $f$ such that $\mathbf{v} = \nabla f$?

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

$$\vec{\nabla} \times \vec{\nabla} f = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix} = \hat{x}\underbrace{\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)}_{=0 \text{ (derivatives commute)}} \left.\begin{aligned} &+ \hat{y} \\ &+ \hat{z} \end{aligned}\right\} \text{same reasoning}$$

If $\vec{v} = \vec{\nabla} f$

then $\vec{\nabla} \times \vec{v} = \vec{\nabla} \times \vec{\nabla} f = 0$ so finding $f$ is possible only if $\vec{\nabla} \times \vec{v} = 0$.

[3.] Given a vector field **v**. Suppose you are asked to compute **A** such that $\mathbf{v} = \nabla \times \mathbf{A}$. What test should you perform on $\mathbf{v}$? Why? Given $\mathbf{v} = x^2\,\hat{x} - z^2\,\hat{y} + (y^2 - 2xz)\,\hat{z}$. Does it past the test? If so, find **A**.

Check that $\vec{\nabla} \cdot \vec{v} = 0$. This is required since $\bar{\nabla} \cdot (\bar{\nabla} \times \bar{A}) = 0$.

$\vec{\nabla} \cdot \vec{v} = 2x + 0 - 2x = 0$ So yes, $\vec{v}$ passes!

the given $\vec{v}$

$$\bar{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$

$\vec{v}$:

$\leftarrow x^2$

$\leftarrow -z^2$

$\leftarrow y^2 - 2xz$

We are free to choose $A_z = 0$. $\Leftarrow$ Could do $A_x = 0$ but this is more simple!

Then $\frac{\partial A_y}{\partial z} = -x^2$ $\quad A_y = -x^2 z + f(x,y)$

$\frac{\partial A_x}{\partial z} = -z^2$ $\quad A_x = -\frac{1}{3} z^3 + g(x,y)$

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -2xz + \frac{\partial f}{\partial x} - \left(0 + \frac{\partial g}{\partial y}\right) = y^2 - 2xz$$

Choose $f = 0$ and $g = \frac{-1}{3} y^3$

$A_x = -\frac{1}{3} z^3 - \frac{1}{3} y^3$

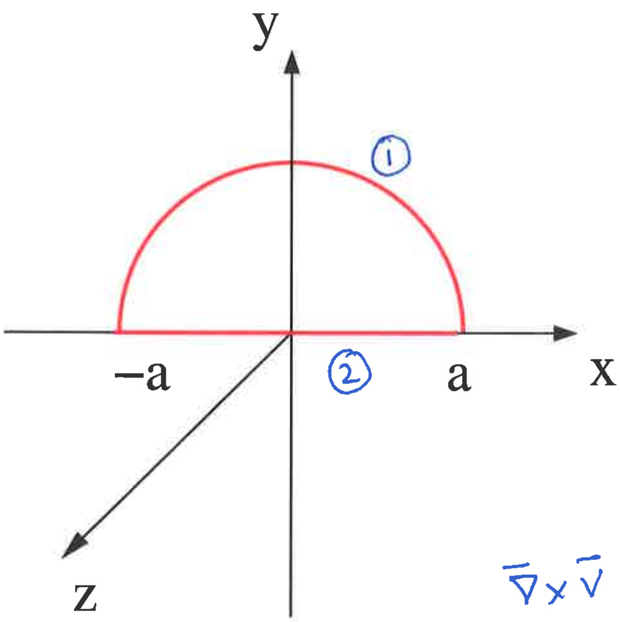
$A_y = -x^2 z$

$A_z = 0$

Check $\vec{\nabla} \times \vec{A} =$

$\left[0 - (-x^2)\right]\hat{x} + \left[-z^2 - 0\right]\hat{y} + \left[-2xz + y^2\right]\hat{z}$

$= x^2\hat{x} - z^2\hat{y} + (y^2 - 2xz)\hat{z}$ ✓✓




on ultimate version of exam, $\int_P \vec{v}\cdot d\vec{l}$ was optional (extra credit).

**[4.]** The vector field $\mathbf{v} = x\,\hat{y}$. Show that $\int_S \nabla\times\mathbf{v}\cdot d\mathbf{a} = \int_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}$ for the figure shown at left, which consists of a closed half circle in the $xy$ plane.

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x & 0 \end{vmatrix} = \hat{z}$$

$$d\vec{a} = dx\,dy\,\hat{z}$$

$$\int_S \vec{\nabla}\times\vec{v}\cdot d\vec{a} = \int_S \underbrace{\hat{z}\cdot\hat{z}}_{1}\,dx\,dy = \int_S dx\,dy = \text{Area} = \tfrac{1}{2}\pi a^2$$

$$d\vec{l} = \hat{x}\,dx + \hat{y}\,dy \qquad \int_{P_1}\vec{v}\cdot d\vec{l} = \int_{P_1} x\hat{y}\cdot(\hat{x}\,dx + \hat{y}\,dy) = \int x\,dy$$

Along ①

$x = \pm\sqrt{a^2-y^2}$

y goes from 0 to a $\leftarrow$ $x = +\sqrt{a^2-y^2}$

and then back from a to ∅ $\leftarrow$ $x = -\sqrt{a^2-y^2}$

} factor of 2

$$\int_{P_1}\vec{v}\cdot d\vec{l} = 2\int_0^a \underbrace{\sqrt{a^2-y^2}}_{x}\,dy = 2\int_0^{\pi/2} a\cos\theta\; a\cos\theta\,d\theta$$

$y = a\sin\theta$

$dy = a\cos\theta\,d\theta$

$$= 2a^2\int_0^{\pi/2}\cos^2\theta\,d\theta = 2a^2\int_0^{\pi/2}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= a^2(\theta + \sin 2\theta)\Big|_0^{\pi/2} = \frac{\pi a^2}{2}$$

Meanwhile along ② $\quad d\vec{l} = \hat{x}\,dx \quad \vec{v}\cdot d\vec{l} = 0 \quad$ so $\int_{P_2}\vec{v}\cdot d\vec{l} = 0$

So get $\frac{1}{2}\pi a^2$ for both integrals together

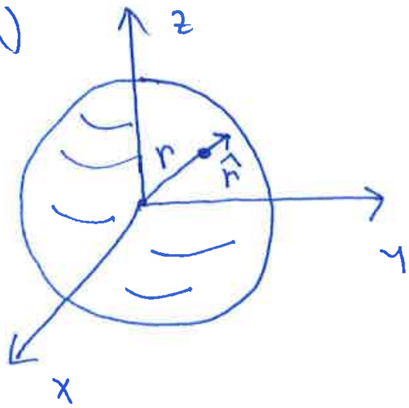
[5.] Consider spherical coordinates.

a) Draw a picture of a surface $r$ =constant. What unit vector is normal to that surface?

b) Draw a picture of a surface $\theta$ =constant. What unit vector is normal to that surface?

c) Draw a picture of a surface $\phi$ =constant. What unit vector is normal to that surface?

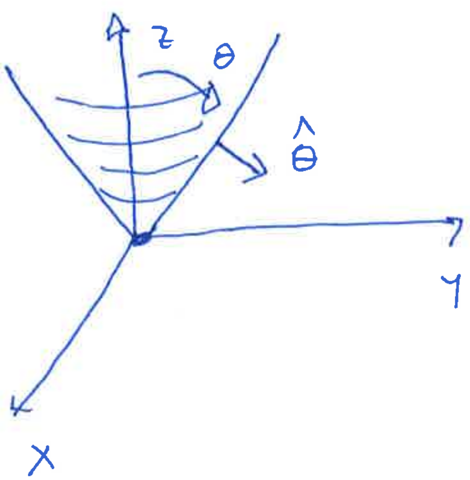
(a)




Surface is Sphere about origin

normal vector is $\hat{r}$

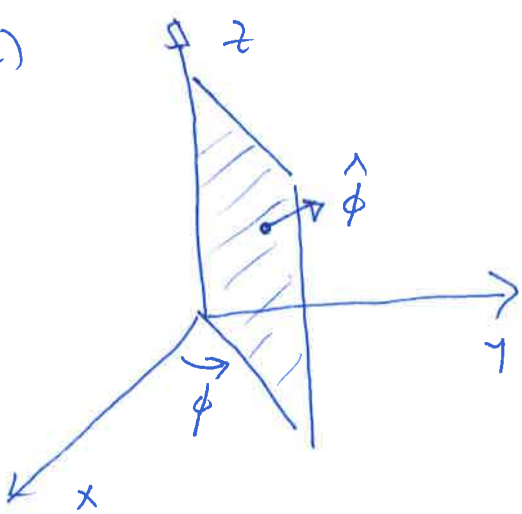
(b)




Surface is a cone with apex at origin

unit vector is $\hat{\theta}$

(c)




Surface is a plane containing $\hat{z}$ axis and making angle $\phi$ with xz plane.

Unit vector is $\hat{\phi}$