KD-1A

Kronecker delta "selects out" particular m value

$n = \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \ldots \quad N$

$x_n = \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \ldots \quad x_N$

$\delta_{nm} = \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \ldots \quad 0$

$m = 3$ nonzero only for $n = m = 3$

$$\sum_{n=1}^{N} x_n \delta_{nm} = 0 \quad 0 \quad x_3 \quad 0 \quad 0 \ldots \quad 0$$

$$\boxed{\sum_{n=1}^{N} x_n \delta_{nm} = x_m}$$

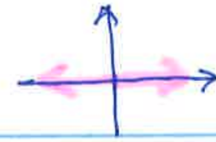
obviously

$$\sum_{n=1}^{N} f(x_n)\, \delta_{nm} = f(x_m)$$

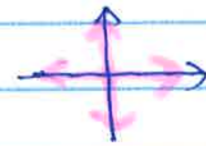
Is there an analogous object which singles things out in a "continuous world"?!

KD-2

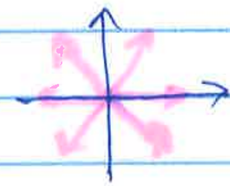
$x^2 = 1 \qquad x = \pm 1$



$x^4 = 1 \qquad x = \pm 1, \pm i$



$x^6 = 1 \qquad x = ?$



General rule:

$$x^N = 1 \qquad x_n = e^{i \frac{2\pi}{N} n} \qquad n = 1, 2, 3 \ldots N$$

Notice that by usual rules of vector addition

$$\sum_{n=1}^{N} x_n = 0$$

One sometimes denotes $k_n = \frac{2\pi n}{N}$

What about $\sum_{n=1}^{N} x_n^2$?

$N = 4 \qquad x_n = \pm 1, \pm i$

$1, 1, -1, -1$

General Rule $\sum_{n=1}^{N} (x_n)^m = 0$ unless $m = 0$ or a multiple of N

$= N$ if $m = 0$

"Umklapp processes" in solid state physics

KD-3

$$\frac{1}{N}\sum_k e^{ikm} = \delta_{m,0}$$

understand
this means
$k = \frac{2\pi}{N} n$
$n = 1, 2, 3, \ldots N$

$$\frac{1}{2\pi}\int dk\, e^{ikx} = \delta(x)$$

unless $x = 0$
the varying phases
in e^{ikx} sum to
zero

where does $1/2\pi$ originate

$$\Delta k = \frac{2\pi}{N} \qquad \frac{1}{N}\sum_k \to \frac{1}{2\pi}\int dk$$

Schroedinger approach
to QM

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

given V ⇝ discrete E

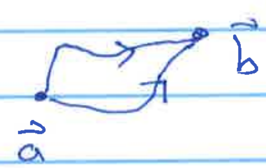
Heisenberg approach
to QM

$\hat{H}$ operator eigenvalues $\to E$

Feynman approach
to QM

$$\int_a^b \mathcal{D}x\, e^{-\frac{i}{\hbar} S_{cl}[x]} = \text{prob}_{a \to b}$$

as $\hbar \to 0$
only path with
$\delta S_{cl}/\delta x = 0$ survives.



we already looked at $\vec{\nabla}\cdot\vec{v}$ with

$$\vec{v} = \frac{c\,\hat{r}}{r^2} = \frac{c\,\vec{r}}{r^3}$$

and concluded, by working in rectangular coordinates, that $\vec{\nabla}\cdot\vec{v} = 0$ except possibly at origin.

verify in spherical coordinates

Much easier than calculations in x, y, z

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}1 = 0$$

except need to remember those funny factors

But look at divergence theorem

$$\int \vec{v}\cdot d\vec{a} = \int c\,R^2 \sin\theta\, d\theta\, d\phi\, \frac{c}{R^2}\,\underbrace{\hat{r}\cdot\hat{r}}_{1} = 4\pi c$$

Sphere radius R about origin

$d\vec{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$

$$\underbrace{\int_0^{\pi}\sin\theta\, d\theta}_{2}\ \underbrace{\int_0^{2\pi}d\phi}_{2\pi}$$

But this should equal

$$\int \vec{\nabla}\cdot\vec{v}\, d\tau$$

?

how could this be if $\vec{\nabla}\cdot\vec{v} = 0$ everywhere?

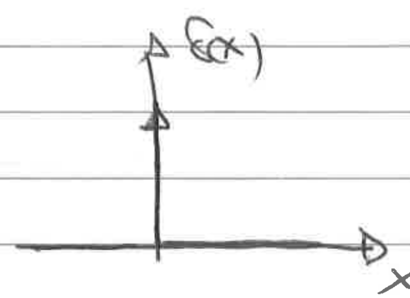
$\vec{\nabla}\cdot\vec{v}$ must be nonzero at origin and also so huge that integration over that single point gives nonzero result.

D2

We give a name to the function that is zero everywhere but so big at a single point that you still get a non-zero value when integrating

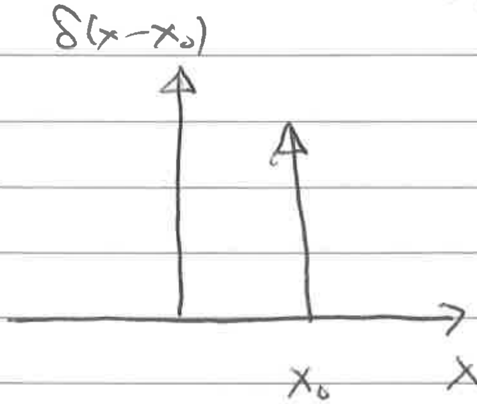
DIRAC DELTA FUNCTION : $\delta(x)$

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \text{"}\infty\text{"} & x = 0 \end{cases}$$



$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x) = f(0)$$



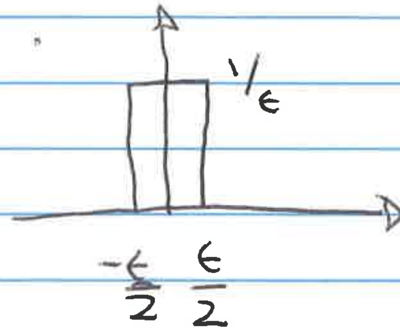
$\delta(x-x_0)$ obviously is nonzero at $x = x_0$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0) = f(x_0)$$

D3.

Think about $\delta(x)$ as limit of family of more conventional functions

$$\delta(x) = \lim_{\epsilon \to 0}$$



Consider $f(x) = 3 - 2x + x^2$

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx \Rightarrow \int_{-\frac{\epsilon}{2}}^{\epsilon/2} f(x)\frac{1}{\epsilon}\,dx$$

$$= \frac{1}{\epsilon}\left(3x - x^2 + \frac{x^3}{3}\right)\Bigg|_{-\epsilon/2}^{\epsilon/2}$$

$$= \frac{1}{\epsilon}\left[3\epsilon - 0 + \frac{\epsilon^3}{12}\right]$$

$$= 3 + \epsilon^2/12 \longrightarrow 3 = f(0)$$

As $\epsilon \to 0$

D4

$$\int_{-\infty}^{\infty} f(x)\,\delta(ax)\,dx \qquad u = ax \qquad x = \frac{u}{a}$$

$$= \int_{-\infty}^{\infty} f\left(\frac{u}{a}\right)\delta(u)\,\frac{du}{a}$$

$$= f(0)\,{}^1\!/_a = \int f(x)\,\frac{1}{a}\,\delta(x)\,dx$$

$$\Longrightarrow \quad \delta(ax) = {}^1\!/_a\,\delta(x)$$

X_α are roots of g:

$g(X_\alpha) = 0$

$$\int_{-\infty}^{\infty} f(x)\,\delta[g(x)]\,dx$$

$$= \sum_\alpha f(x_\alpha)\,\frac{1}{|g'(x_\alpha)|}$$

Encounter in Stat mech course

in computing "density of states":

Given dispersion relation $E(\vec{k})$ p

How many states have given E

$$N(E) \equiv \int d^3k\,\delta\left(E - \epsilon(\vec{k})\right)$$

p p

DS

$$\epsilon_p = \frac{p^2}{2m} = \frac{p_x^2 + p_y^2 + p_z^2}{2m}$$

$$N(E) = \int dp_x\, dp_y\, dp_z\, \delta(E - \epsilon_p)$$

polar coordinates for $\vec{p}$ just like $\vec{r}$!

$$= \underbrace{\int_0^{\pi} d\theta \int_0^{2\pi} d\phi}_{4\pi} \int_0^{\infty} p^2\, dp\, \delta(E - p^2/2m)$$

$g(p) = E - p^2/2m$

$g'(p) = -p/m$

$E - p^2/2m = 0$

$p = \sqrt{2mE}$

$$N(E) = \underbrace{\left(\sqrt{2mE}\right)^2}_{f(p_\alpha)} \underbrace{m/\sqrt{2mE}}_{|1/g'(p_\alpha)|}$$

$N(E) \sim E^{1/2}$ ← a famous result in SSP

$\Rightarrow C(T) = aT$ e^- in metal

Compare to $C(T) = \frac{3}{2}k_B = T$ indep

classical ideal gas