DI-0

We have focussed thus far on

(1) Differentiation

$\vec{\nabla}\phi \qquad \vec{\nabla}\cdot\vec{V} \qquad \vec{\nabla}\times\vec{V}$

and second derivatives; some messy

others remarkably simple

$$\left\{ \begin{array}{l} \vec{\nabla}\times\vec{\nabla}\phi = 0 \\ \vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = 0 \end{array} \right.$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{V}) - \nabla^2 V$$

$$\vec{\nabla}\cdot\vec{\nabla}\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

(2) Integration in sense of

$$\int_{\vec{a}}^{\vec{b}} \vec{\nabla}\phi\cdot d\vec{l} = \phi(\vec{b}) - \phi(\vec{a})$$

$$\int_{V} \vec{\nabla}\cdot\vec{V}\, d\tau = \int_{S} \vec{V}\cdot d\vec{a}$$

more traditional meaning of integration

$$\int_{S} \vec{\nabla}\times\vec{V}\cdot da = \int_{P} \vec{V}\cdot d\vec{l}$$

What about finding functions given their derivatives

DI-1

Fundamental Theorem of Calculus

Differentiation and integration are inverses

This process is much harder than differentiation ①

$$\frac{d}{dx}\frac{1}{\sqrt{1-x^2}} = \frac{d}{dx}(1-x^2)^{-1/2} = -\tfrac{1}{2}(1-x^2)^{-3/2}(-2x) \quad W$$

$\int \frac{dx\, x}{(1-x^2)^{3/2}} = \frac{1}{(1-x^2)^{1/2}}$

$$\int \frac{dx}{\sqrt{1-x^2}} = ?$$ What function has derivative $\frac{1}{\sqrt{1-x^2}}$

answer is $\sin^{-1} x + C$

So not obvious at all

and also the answer is not unique. ②

Similar for vector differentiation and integration

$$\phi(x,y,z) = x y^2 z^3$$

Simple to get $\vec{V} = \vec{\nabla}\phi = \underbrace{y^2 z^3}_{\partial\phi/\partial x}\hat{x} + \underbrace{2xyz^3}_{\partial\phi/\partial y}\hat{y} + \underbrace{3xy^2z^2}_{\partial\phi/\partial z}\hat{z}$

But what if we are given $\vec{V}$. How to get ϕ?

DI-2

In fact it is not even assured that ϕ exists

Vector identity $\vec{\nabla} \times \vec{\nabla} \phi = 0$

Thus in order for ϕ to exist $\vec{\nabla} \times \vec{v} = 0$

So if you are given $\vec{v}$ and asked to compute ϕ

Ask / check $\vec{\nabla} \times \vec{v} = 0$ ← if not impossible

all equivalent

$$\left\{ \begin{array}{l} \vec{\nabla} \times \vec{v} = 0 \\ \displaystyle\int_a^b \vec{v} \cdot d\vec{\ell} = \text{path independent} = \phi(\vec{b}) - \phi(\vec{a}) \\ \displaystyle\oint \vec{v} \cdot d\vec{\ell} = 0 \\ \vec{v} \text{ is conservative} \\ v = \nabla \phi \end{array} \right.$$

arises in Classical Mechanics

ϕ ← potential

$\vec{v}$ ← Force

$\int_a^b \vec{F} \cdot d\vec{\ell}$ = Work done

actually

$\vec{F} = -\vec{\nabla} \phi$

EM $\vec{\nabla} \times \vec{E} = 0$

$-\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l}$ = path independent $= V(\vec{b}) - V(\vec{a})$

As in CM

$\oint \vec{E} \cdot d\vec{l} = 0$

$\vec{E} = -\vec{\nabla}\phi$

[Actually is $\oint \vec{E} \cdot d\vec{l} = 0$ always? Yes if time independent field: Electrostatics

$\frac{d}{dt}\phi_B$

Faraday's law

$\vec{\nabla} \times \vec{E} \neq 0$ if $\frac{d\vec{B}}{dt} \neq 0$]

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such that $\vec{v} = \vec{\nabla}\phi$

So, how do we get ϕ from $\vec{v}$

(1) Check $\vec{\nabla} \times \vec{v} = 0$!

(2) Choose an arbitrary location $\vec{a}$ where $\phi = 0$

like the arbitrary constant in conventional integration

$$\phi(\vec{r}) - \phi(\vec{a}) = \int_a^{\vec{r}} \vec{v} \cdot d\vec{\ell}$$

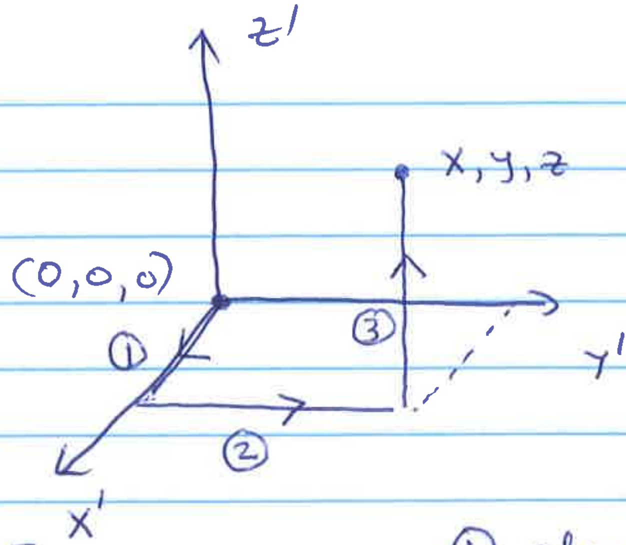
$\vec{v} = \vec{\nabla}\phi$

path doesn't matter so just pick one

$$\vec{v} = (3x^2yz - 3y)\hat{x} + (x^3z - 3x)\hat{y} + (x^3y + 2z)\hat{z}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2yz - 3y & x^3z - 3x & x^3y + 2z \end{vmatrix} = \hat{x}(x^3 - x^3) + \hat{y}(3x^2y - 3x^2y) + \hat{z}(3x^2z - 3 - 3x^2z + 3) = \phi$$

DI-5



primes to distinguish integration variable from endpoint

choose $\vec{a} = (0,0,0)$

① along path $y' = z' = 0$

$$d\vec{l} = \hat{x}\, dx$$

$$\int \vec{v} \cdot d\vec{l} = \int (3x'^2 y' - 3y')\, dx' = 0$$

② along path $z' = 0 \quad x' = x$

$$d\vec{l} = \hat{y}\, dy$$

$$\int \vec{v} \cdot d\vec{l} = \int_0^y (\underbrace{x'^3 z'}_{0} - 3\underbrace{x'}_{x})\, dy'$$

$$= -3x \int_0^y dy' = -3xy$$

③ along path

$x' = x \quad y' = y$

$d\vec{l} = \hat{z}\, dz$

$$\int_0^z (x'^3 y' + 2z')\, dz'$$

$$= x^3 y \int_0^z dz' + 2\int_0^z z'\, dz'$$

$$= x^3 y z + z^2$$

$$\phi(x,y,z) = -3xy + x^3 y z + z^2$$

check

$$\frac{\partial \phi}{\partial x} = -3y + 3x^2 y z$$

$$\frac{\partial \phi}{\partial y} = -3x + x^3 z$$

$$\frac{\partial \phi}{\partial z} = x^3 y + 2z$$

DI-7

$\vec F = \vec\nabla \phi \qquad \phi \leftarrow$ arbitrary constant

Fortunately freedom is even more rich in

$\vec V = \vec\nabla \times \vec A$ "gauge freedom"

very deep concept in physics

EM photons

particles which mediate interactions

Weak interactions W, Z bosons

strong interactions $\underbrace{q \qquad q}_{\text{quarks}}$ "gluons"

Can choose $A_x = 0$ (or A_y, A_z)

Given $\vec V = (x^2 - yz)\hat x - 2yz\,\hat y + (z^2 - 2xz)\hat z$

Find $\vec A$ such that $\vec V = \vec\nabla \times \vec A$

First check $\vec\nabla \cdot \vec V = 0$

$$\underbrace{2x}_{\frac{\partial V_x}{\partial x}} \underbrace{- 2z}_{\frac{\partial V_y}{\partial y}} + \underbrace{2z - 2x}_{\frac{\partial V_z}{\partial z}} = 0 \quad \checkmark\checkmark$$

DI-8

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = v_x = x^2 - yz$$

$$\cancel{\frac{\partial A_x}{\partial z}} - \frac{\partial A_z}{\partial x} = v_y = -2yz$$

$$\phi \quad \frac{\partial A_y}{\partial x} - \cancel{\frac{\partial A_x}{\partial y}} = v_z = z^2 - 2xz$$

"gauge freedom"

$$\phi$$

$$\frac{\partial A_y}{\partial x} = z^2 - 2xz \implies A_y = z^2 x - x^2 z + f(y,z)$$

$$\frac{\partial A_z}{\partial x} = 2yz \implies A_z = 2xyz + g(y,z)$$

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 2xz + \frac{\partial g}{\partial y} - 2zx + x^2 - \frac{\partial f}{\partial z}$$

$$= x^2 - yz$$

$$\frac{\partial g}{\partial y} - \frac{\partial f}{\partial z} = -yz$$

still a lot of freedom!

eg $\quad g = -\frac{1}{2}y^2 z \quad f = 0$

or $\quad g = 0 \quad f = \frac{1}{2}yz^2 \quad$ etc

DI-9

$A_x = 0$

$A_y = z^2 x - x^2 z$

$A_z = 2xyz - \frac{1}{2} y^2 z$

check $\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 2xz - yz - 2xz + x^2$

$= x^2 - yz$ ✓✓

$= V_x$

etc...