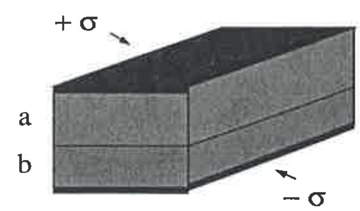
KEY

PHYSICS 110A, WINTER 2018
ELECTRICITY AND MAGNETISM

Final Exam

[1.] The space between a parallel plate capacitor is filled with two slabs of linear dielectric material. The top slab has thickness a and dielectric constant 2.0. The bottom slab has thickness b and dielectric constant 3.0, The free charge on the top plate is $+\sigma$ and on the bottom plate is $-\sigma$.

(a) Find the electric displacement **D** in each slab.
(b) Find the electric field **E** in each slab.
(c) Find the polarization **P** in each slab.
(d) Find the potential difference between the plates.
(e) Find the location and amount of all bound charge.



$\uparrow \hat{z}$

(a) $\vec{D}$ only responds to free charge, so $|\vec{D}| = \sigma/\epsilon_0$. This results from contributions of $\sigma/2\epsilon_0$ from top and bottom which add to σ/ϵ_0. $\vec{D}$ in $-\hat{z}$ direction.

(b) $|\vec{E}| = |\vec{D}|/\epsilon$ so $|\vec{E}| = \sigma/2\epsilon_0$ in the top slab, and $|\vec{E}| = \sigma/3\epsilon_0$ in the bottom slab. $\vec{E}$ in $-\hat{z}$ direction

(c) $\vec{P} = \vec{D} - \epsilon_0\vec{E} \Rightarrow |\vec{P}| = \sigma/\epsilon_0 - \epsilon_0(\sigma/2\epsilon_0) = \sigma/2\epsilon_0$ (top)

$\vec{P}$ in $-\hat{z}$ direction $\quad = \sigma/\epsilon_0 - \epsilon_0(\sigma/3\epsilon_0) = 2\sigma/3\epsilon_0$ (bottom)

(d) $$V = \int E\,d\ell = \underbrace{\frac{\sigma}{3\epsilon_0}b}_{\text{bottom}} + \underbrace{\frac{\sigma}{2\epsilon_0}a}_{\text{top}} = \frac{\sigma}{\epsilon_0}\left(\frac{b}{3} + \frac{a}{2}\right) = \frac{\sigma}{6\epsilon_0}(2b + 3a)$$

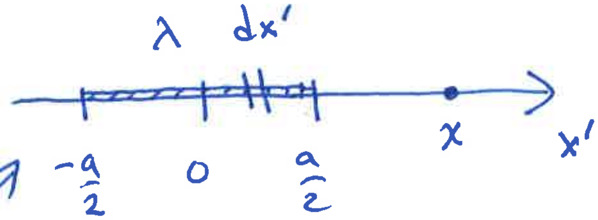
(e) $\sigma_b = \vec{P}\cdot\hat{n} \Rightarrow$ top of top slab $\vec{P}\downarrow$ $\hat{n}\uparrow \Rightarrow \vec{P}\cdot\hat{n} = -\sigma/2\epsilon_0$

bottom " " " $\vec{P}\downarrow$ $\hat{n}\downarrow \Rightarrow \vec{P}\cdot\hat{n} = +\sigma/2\epsilon_0$

top of bottom slab $\vec{P}\downarrow$ $\hat{n}\uparrow \Rightarrow \vec{P}\cdot\hat{n} = -\frac{2\sigma}{3\epsilon_0}$

bottom " " " $\vec{P}\downarrow$ $\hat{n}\downarrow \Rightarrow \vec{P}\cdot\hat{n} = +\frac{2\sigma}{3\epsilon_0}$

[2.] Consider a thin wire of length a and uniform linear charge density λ. Compute the electric field $\mathbf{E}$ along the axis of the wire. Consider both the case when you are evaluating $\mathbf{E}$ at points outside the wire, and for points inside the wire.



$0 = E_y = E_z$ along wire axis

$x > \frac{a}{2}$ i.e. outside the wire

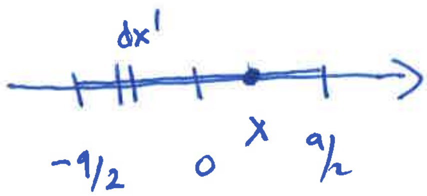
$$E_x = \int_{-a/2}^{a/2} \frac{1}{4\pi\epsilon_0} \frac{\lambda\, dx'}{(x-x')^2} = \frac{\lambda}{4\pi\epsilon_0} \frac{1}{x-x'}\Big|_{-a/2}^{a/2}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left\{\frac{1}{x-a/2} - \frac{1}{x+a/2}\right\} = \frac{\lambda a}{4\pi\epsilon_0} \frac{1}{x^2 - a^2/4}$$

← has correct point charge limit for $x \gg a$

$x < a/2$ ie inside the wire

$$E_x = \int_{-a/2}^{x-\delta} \frac{1}{4\pi\epsilon_0} \frac{\lambda\, dx'}{(x-x')^2} - \int_{x+\delta}^{a/2} \frac{1}{4\pi\epsilon_0} \frac{\lambda\, dx'}{(x-x')^2}$$



excise small region about x and later take limit $\delta \to 0$

because portion of wire with $x' > x$ gives $\vec{E}$ field to the left.

$$= \frac{\lambda}{4\pi\epsilon_0}\left\{ \frac{1}{x-x'}\Big|_{-a/2}^{x-\delta} - \frac{1}{x-x'}\Big|_{x+\delta}^{a/2} \right\}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left\{ \left(\frac{1}{\delta} - \frac{1}{x+a/2}\right) - \left(\frac{1}{x-a/2} + \frac{1}{\delta}\right)\right\}$$

← cancels other $+1/\delta$ term

$$= \frac{\lambda}{4\pi\epsilon_0}\, 2x \Big/ (a^2/4 - x^2)$$

← vanishes at $x = 0$ ✓✓

← dimensionally correct

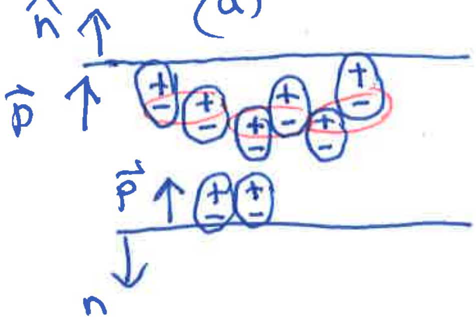
[3.] The energy U of a collection of point charges is $U = \frac{1}{2}\sum_{i,j} q_i q_j / r_{ij}$ where r_{ij} is the distance between charges q_i and q_j. It is possible that $U < 0$ if one has charges of opposite sign. On the other hand, we also have an expression $U = \frac{1}{2}\epsilon_0 \int |E|^2 d\tau$, which is always positive. In just a few sentences, discuss the resolution of this seeming paradox.

The expression $\frac{1}{2}\epsilon_0 \int |E|^2 d\tau$ includes the energy to assemble the individual point charges. This large + contribution forces even a $q_1 q_2 / r_{12} < 0$ contribution (which doesn't include the cost of assembling q_1 and q_2) to be positive.

[4.] (a) A huge slab $-\infty < x < +\infty$, $-\infty < y < +\infty$, $-a < z < +a$, of polarizable material is placed in an external electric field $\mathbf{E} = E_0\,\hat{\mathbf{z}}$. A polarization $\mathbf{P} = \chi_E \mathbf{E}$ results. The material is uniform, i.e. χ_E is the same throughout the slab. Draw a picture and write a few sentences which provide an argument for why there is charge on the surfaces, but not in the interior. Explain how the signs of the charges on the surfaces are consistent with $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$.

(b) Now suppose the material is not uniform, so that $\chi_E = \chi_{E0}\,(z+a)/(2a)$ vanishes at $z = -a$ and increases to χ_{E0} at $z = +a$. Explain physically, with an appropriate picture, why in this case there is a non-vanishing volume charge density ρ_b. What does the formula for ρ_b give?

(a)



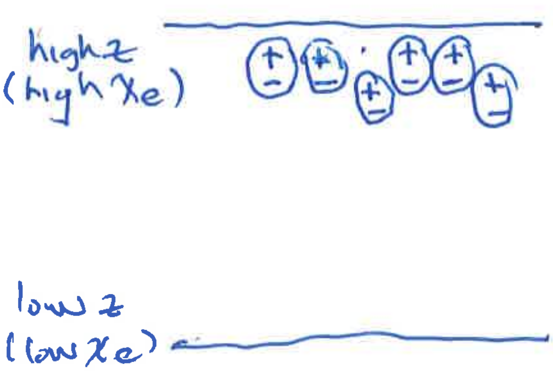
← + charge at surface $\vec{P}\cdot\hat{n} > 0$

← inside + and − cancel

⇒ no volume charge inside

← − charge at surface $\vec{P}\cdot\hat{n} < 0$

(b) If material is more polarizable at larger z then we should draw more dipoles there



⇐ more dipoles as we get to higher z

Cancellation in interior is now incomplete and $\rho_b \neq 0$

$$\vec{\nabla}\cdot\vec{P} = \frac{\partial}{\partial z} P_z = \frac{\partial}{\partial z} \underbrace{\chi_{E0}\frac{z+a}{2a}}_{\chi_e} \underbrace{E_0}_{E_z}$$

$$= \frac{1}{2}\chi_{E0} E_0$$ ← note sign is correct!

[5.] Charge q_1 is located at position $(a,0,0)$; Charge q_2 is located at position $(0,b,0)$; and Charge q_3 is located at position $(0,0,c)$. What is the simplest thing you can say about the potential V at positions a distance r from the charges if $r \gg a,b,c$?

For $r \gg a,b,c$ only the monopole term of V survives

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q_1+q_2+q_3}{r}$$

[6.] A hydrogen atom acts as if it had an electrostatic potential

$$\Phi(r) = \frac{q}{4\pi\epsilon_0 r}(1+\frac{r}{a_0})e^{-2r/a_0},$$

where q is the charge on the proton and $a_0 = \hbar^2/m_e q^2 = 0.529\AA$ is the Bohr radius. Find the corresponding charge density and interpret the various terms physically. Hint: The Laplacian in spherical coordinates is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\left.\begin{aligned}\vec\nabla\cdot\vec E &= \rho/\epsilon_0\\ \vec E &= -\vec\nabla V\end{aligned}\right\} \rightarrow \nabla^2 V = -\rho/\epsilon_0$$

$$\rho(r) = -\epsilon_0\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right)\frac{q}{4\pi\epsilon_0}\left(\frac{1}{r}+\frac{1}{a_0}\right)e^{-2r/a_0}$$ ← no θ,ϕ dependence

We can do derivatives in usual way except at $\vec r = 0$, because $1/r$ factors diverge. We need to recall that $\nabla^2 1/r = -4\pi\delta(\vec r)$

At origin $r/a_0 = 0$ and $e^{-2r/a_0} = 1$ so

$$\rho(r) = -\epsilon_0\frac{q}{4\pi\epsilon_0}(-4\pi\delta(r)) - \frac{q}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\left(-\frac{2}{a_0 r}-\frac{1}{r^2}-\frac{2}{a_0^2}\right)e^{-\frac{2r}{a_0}}$$

$$= +q\delta(r) - \frac{q}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(-\frac{2r}{a_0}-1-\frac{2r^2}{a_0^2}\right)e^{-2r/a_0}$$

the proton!

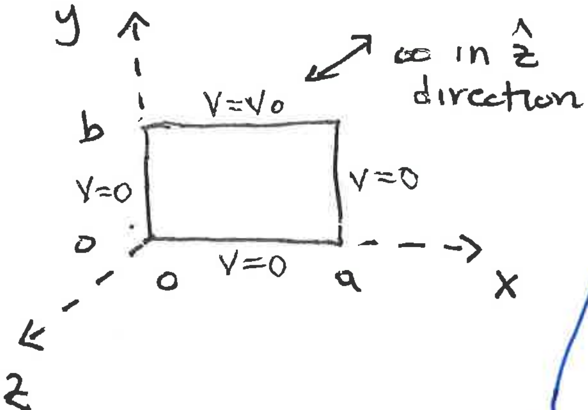
$$-\frac{q}{4\pi r^2}\left\{-\frac{2}{a_0}-\frac{4r}{a_0^2}-\frac{2}{a_0}\left(-\frac{2r}{a_0}-1-\frac{2r^2}{a_0^2}\right)\right\}e^{-2r/a_0}$$

$$-\,q/4\pi r^2\left(\frac{4r^2}{a_0^3}\right)e^{-2r/a_0}$$

the e^-!

$$-\frac{q}{\pi a_0^3}e^{-2r/a_0}$$

[7.] An infinitely long rectangular tube $0 < x < +a$, $0 < y < +b$, and $-\infty < z < +\infty$, has potential $V = 0$ for $x = 0$; $V = 0$ for $x = a$; $V = 0$ for $y = 0$; and $V = V_0$ for $y = b$. (See Figure.) Solve Laplace's equation for the potential in the interior.



Separation of variables for Laplace Eqn

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) V(x,y) = \not{\rho}$$

no charge inside!

no z dependence because tube is ∞ long!

$$V(x,y) = f(x)\, g(y)$$

$$g(y) \frac{d^2 f}{dx^2} + f(x) \frac{d^2 g}{dy^2} = 0$$

$$\frac{1}{f} \frac{d^2 f}{dx^2} = \frac{1}{g} \frac{d^2 g}{dy^2} = -k^2$$

$f(x) = \sin kx$; $\cos kx$ $\rightsquigarrow$ $\sin kx$ only since $V=0$ at $x=0$

$g(y) = \sinh ky$; $\cosh ky$ $\rightsquigarrow$ $\sinh ky$ only since $V=0$ at $y=0$

finally $k = \frac{n\pi}{a}$ since $V=0$ at $x=a$

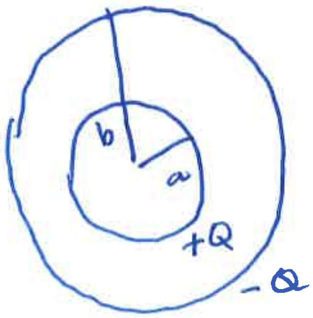
$$V(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

$$V(x, y=b) = V_0 = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{a} \sinh n\pi b/a$$

Multiply both sides by $\sin m\pi x/a$ and integrate $\int_0^a dx$

use orthogonality! Eqn for a_m $\longrightarrow$ $\int_0^a V_0 \sin\frac{m\pi x}{a} dx = a_m \frac{a}{2} \sinh\frac{m\pi b}{a}$

[8.] Compute the capacitance of a system of two concentric metallic spherical shells of inner radius a and outer radius b.



By Gauss' law $E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$ for $a<r<b$

$$V = \int_a^b \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,dr$$

$$= \frac{Q}{4\pi\epsilon_0 r}\Big|_a^b = -\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{b}-\frac{1}{a}\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{b-a}{ab}\right)$$

Now $C = \frac{Q}{V} = 4\pi\epsilon_0\, ab/_{b-a}$

check limit $\underbrace{b = a + d}_{d \text{ small } \#} \Rightarrow ab \gtrsim a^2$

$$C = \epsilon_0 \frac{4\pi a^2}{d} = \frac{\epsilon_0 A}{d}$$

$A = 4\pi a^2$ = area of sphere

expected result for parallel plate capacitor

Do only $\boxed{\text{one}}$ of problems 9, 10.

[9.] A *metallic* sphere of radius R is placed in a uniform electric field $\mathbf{E}_{\rm ext} = E_0\,\hat{\mathbf{z}}$. Compute the potential everywhere in space.

① $V = $ const on sphere surface and $V=0$ on equator by symmetry $\Rightarrow$ $V=0$ on sphere surface

② $V \to -E_0 r\cos\theta$ at $r \to \infty$

$$V(r,\theta) = \sum_{l=1}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right]P_l(\cos\theta)$$

$$V(R,\theta) = \sum_{1}^{\infty}\left[A_l R^l + B_l R^{-(l+1)}\right]P_l(\cos\theta) = 0$$

For this to be true for all θ need $B_l = -A_l R^{2l+1}$

$$V(r,\theta) = \sum_{1}^{\infty} A_l\left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right)P_l(\cos\theta)$$

But for $V \to -E_0 r\cos\theta$ must have $A_l = 0$ $l \neq 1$ and $E_1 = -E_0$

$r > R$ $\quad V(r,\theta) = -E_0\left(r - R^3/r^2\right)\cos\theta$

$r < R$ $\quad V = $ value at surface since metallic

$V = 0$ inside

Do only $\boxed{\text{one}}$ of problems 9, 10.

[10.] A sphere of dielectric constant ϵ is placed in a uniform external electric field $\mathbf{E}_{\rm ext} = E_0\,\hat{\mathbf{z}}$. Compute the potential everywhere in space, as well as the volume(surface) polarization charges within(on) the sphere.

① $V_{in} = V_{out}$ at $r = R$ $\leftarrow$ continuity of V

② $\epsilon \dfrac{\partial V_{in}}{\partial r} = \epsilon_0 \dfrac{\partial V_{out}}{\partial r}$ at $r = R$ $\leftarrow$ continuity of D_n (no free charge) on sphere surface

③ $V_{out} \to -E_0 r \cos\theta$ at $r = \infty$

$$V_{in} = \sum A_\ell r^\ell P_\ell(\cos\theta)$$

$$V_{out} = \sum B_\ell r^{-(\ell+1)} P_\ell(\cos\theta) - E_0 r \cos\theta$$

① Continuity of V at $r = R$ $\quad \ell \neq 1 \quad A_\ell R^\ell = B_\ell R^{-(\ell+1)}$

$\ell = 1 \quad A_1 R = B_1 R^{-2} - E_0 R$

② Continuity of D_n at $r = R$

$\ell \neq 1 \qquad \epsilon/\epsilon_0 \, \ell A_\ell R^{\ell-1} = -(\ell+1) B_\ell R^{-(\ell+2)}$

$\ell = 1 \qquad \epsilon/\epsilon_0 \, A_1 = -E_0 - 2B_1/R^3$

For $\ell \neq 1$ we must have $A_\ell = B_\ell = 0$

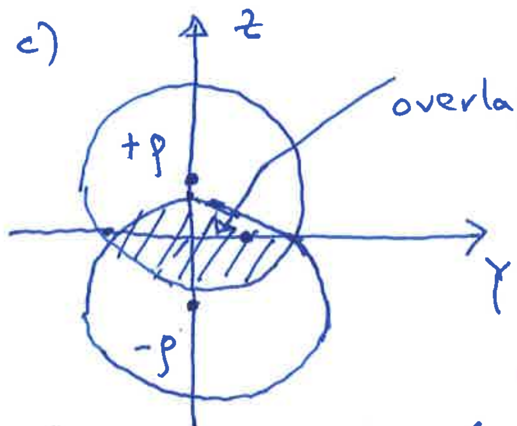
For $\ell = 1 \qquad A_1 = -\dfrac{3}{\frac{\epsilon}{\epsilon_0}+2} E_0 \qquad B_1 = \dfrac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2} R^3 E_0$

[11.] (a) Derive, from Gauss' Law, the *magnitude* of the electric field $|\mathbf{E}|$ at a distance $r < R$ from the center of a sphere of constant charge density $+\rho$ and radius R.
(b) If you place the center at the origin $(0, 0, 0)$ write an expression for the vector $\mathbf{E}$ in terms of the Cartesian coordinate unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. (That is, find the *direction* of $\mathbf{E}$ along with the magnitude from part (a).
(c) Now suppose you have two spheres, one with constant charge density $+\rho$ and center at $(0, 0, +d/2)$ and the other with constant charge density $-\rho$ and center at $(0, 0, -d/2)$. Derive the electric field in the region of overlap. The final answer is surprisingly simple!

a) $$\underbrace{4\pi r^2 |\vec{E}|}_{\Phi_E} = Q/\epsilon_0 = \tfrac{4}{3}\pi r^3 \rho/\epsilon_0 \Rightarrow |\vec{E}| = \frac{r}{\epsilon_0}\rho$$

b) $$\vec{E} = \frac{r}{\epsilon_0}\rho\left(\frac{x\hat{x} + y\hat{y} + z\hat{z}}{r}\right) = \frac{\rho}{\epsilon_0}(x\hat{x} + y\hat{y} + z\hat{z})$$

c)

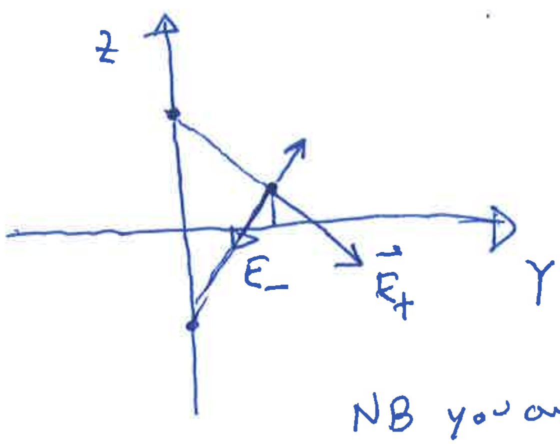


overlap region.

Using (b) at point P with z=0

get signs from picture

$$\vec{E} = \frac{\rho}{\epsilon_0}\left(0\hat{x} + y\hat{y} - \left(\frac{d}{2} - z\right)\hat{z}\right)$$
$$+ \rho/\epsilon_0\left(0\hat{x} - y\hat{y} - \left(\frac{d}{2} + z\right)\hat{z}\right)$$



$$\vec{E} = \rho/\epsilon_0\, \hat{z}$$

$\vec{E}$ is independent of where you are located in the overlap region!!

NB you are free to choose axes so that point of computation of $\vec{E}$ is in yz plane because of azimuthal symmetry