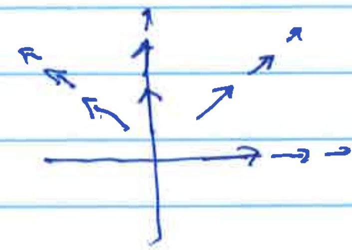
V9A

Point charge

unit vector radially outward

$$\vec{E} = \frac{kQ}{r^2}\hat{r} \qquad \text{Coulomb's law}$$



In class just use "A" for kQ and $\vec{v}$ for $\vec{E}$

$$\frac{x\hat{x} + y\hat{y} + z\hat{z} = \vec{r}}{\sqrt{x^2+y^2+z^2} = r}$$

$$\vec{E} = \frac{kQ}{r^3}\left(x\hat{x} + y\hat{y} + z\hat{z}\right)$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{1/2} = \frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2} 2x = x/r$$

Two pieces $\quad \frac{kQ}{r^3} - \frac{3kQx}{r^4}\frac{\partial r}{\partial x}$

$$\frac{kQ}{r^3} - \frac{3kQx^2}{r^5}$$

analogous for $\partial E_y/\partial y$ and $\partial E_z/\partial z$

$$\frac{3kQ}{r^3} - \frac{3kQ}{r^5}\left(x^2+y^2+z^2\right) = \phi\ !$$

$\vec{\nabla}\cdot\vec{E} = 0$ everywhere (except at $\vec{r} = 0$ where our calculation is suspect!)

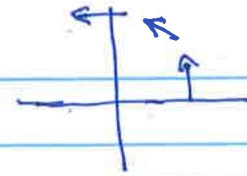
Homework $\quad \vec{E} \sim c\, r\, \hat{r} \quad$ will find $\vec{\nabla}\cdot\vec{E} \neq 0$ even away from origin

<u>Significance</u> $\quad \vec{\nabla}\cdot\vec{E} \sim$ charge density

point charge vs sphere of charge.

V9B

Analogous calculation for $\vec{\nabla}\times$



$$\vec{V} = A\,\frac{1}{\rho^2}\left(x\hat{y} - y\hat{x}\right) \qquad \rho = \sqrt{x^2+y^2}$$

$$\begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{array}$$

$$(\vec{\nabla}\times\vec{V})_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} = 0 \quad \text{(obviously)}$$

$$(\vec{\nabla}\times\vec{V})_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} = 0 \quad \text{(obviously)}$$

$$(\vec{\nabla}\times\vec{V})_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$$

$\partial\rho/\partial x$

$$\frac{\partial}{\partial x}\frac{Ax}{\rho^2} = \frac{A}{\rho^2} - \frac{2Ax}{\rho^3}\,\frac{x}{\rho} = \frac{A}{\rho^2} - \frac{2Ax^2}{\rho^4}$$

$$\frac{\partial v_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-Ay}{\rho^2}\right) = -\frac{A}{\rho^2} + \frac{2Ay^2}{\rho^4}$$

$$\partial v_y/\partial x - \partial v_x/\partial y = \frac{2A}{\rho^2} - \frac{2A}{\rho^4}(x^2+y^2) = 0$$

Significance: $\vec{V} \to \vec{B}$

$\vec{\nabla}\times\vec{B} \sim$ current density

Homework $\vec{V}$ appropriate for current carrying wire of finite thickness

V10

What happens when you act with $\vec{\nabla}$ in various ways on sums and products? That is, what are the analogs of

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + dg/dx$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g \qquad ?$$

$$\vec{\nabla}(f+g) = \vec{\nabla}f + \vec{\nabla}g$$

$$\vec{\nabla}\cdot(\vec{A}+\vec{B}) = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{B}$$

$$\vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B}$$

But all is not quite so simple

$$\vec{\nabla}(fg) = f\vec{\nabla}g + g\vec{\nabla}f$$

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$$

vector! so answer cannot be $\vec{B}\cdot\vec{\nabla}A$ and $\vec{B}\vec{\nabla}A$ makes no sense

What a mess!

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot\vec{\nabla}\times\vec{B}$$

VII

Let's try examples first

Skip: Give as HW!

$$\vec{A} = x\hat{x} - 2y\hat{y} + 3x\hat{z}$$

$$\vec{B} = x^2 z\hat{x} - z^2\hat{y}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & -2y & 3x \\ x^2 z & -z^2 & 0 \end{vmatrix} = \hat{x}(3xz^2) + \hat{y}(3x^3 z) + \hat{z}(-xz^2 + 2yx^2 z)$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = \frac{\partial}{\partial x}(3xz^2) + \frac{\partial}{\partial y}(3x^3 z) + \frac{\partial}{\partial z}(-xz^2 + 2yx^2 z)$$

$$= 3z^2 + 0 - 2xz + 2yx^2$$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & -2y & 3x \end{vmatrix}$$

$$= \hat{x}(0) + \hat{y}(-3) + \hat{z}(0)$$

$$= -3\hat{y}$$

$$\vec{\nabla}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 z & -z^2 & 0 \end{vmatrix}$$

$$= \hat{x}(2z) + \hat{y}(x^2) + \hat{z}(0)$$

$$= 2z\hat{x} + x^2\hat{y}$$

$$\vec{B}\cdot\vec{\nabla}\times\vec{A} = +3z^2$$

$$\vec{A}\cdot(\vec{\nabla}\times\vec{B}) = 2xz - 2yx^2$$

$$\vec{B}\cdot\vec{\nabla}\times\vec{A} - \vec{A}\cdot(\vec{\nabla}\times\vec{B}) = 3z^2 - 2xz + 2yx^2$$

V12

Formal proof

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{A} \times \vec{B}$$

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B})$$

$$= \frac{\partial}{\partial x}(A_y B_z - A_z B_y) - \frac{\partial}{\partial y}(A_x B_z - A_z B_x) + \frac{\partial}{\partial z}(A_x B_y - A_y B_x)$$

look at B_x terms:

$$= B_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right)$$

$$= B_x (\vec{\nabla} \times \vec{A})_x$$

one piece of

$$\vec{B} \cdot (\vec{\nabla} \times \vec{A})$$

V12 A

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \frac{\partial}{\partial x_i}(A\times B)_i$$

$$= \frac{\partial}{\partial x_i}\epsilon_{ijk}A_jB_k$$

$$= \epsilon_{ijk}\frac{\partial A_j}{\partial x_i}B_k + \epsilon_{ijk}A_j\frac{\partial B_k}{\partial x_i}$$

$$(\vec{\nabla}\times\vec{A})_k = \epsilon_{kij}\frac{\partial}{\partial x_i}A_j$$

$$(\nabla\times B)_j = \epsilon_{jik}\frac{\partial B_k}{\partial x_i}$$

$\epsilon_{ijk} = \epsilon_{kij}$

$\epsilon_{jik} = -\epsilon_{ijk}$

$$= (\vec{\nabla}\times\vec{A})_k B_k - (\nabla\times B)_j A_j$$

$$= \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

V13

Use $g(x,y,z)$ for (1)+(2) not f.

Just as in classical mechanics... one needs to go to 2nd, but no higher.

Second derivatives

(1) $\vec{\nabla}\cdot(\vec{\nabla} f) = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial}{\partial y}\frac{\partial f}{\partial y} + \frac{\partial}{\partial z}\frac{\partial f}{\partial z}$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \equiv \nabla^2 f$$

"Laplacian"

In contrast to complexity, some simplicity...

(2) $\vec{\nabla}\times(\vec{\nabla} f) = 0$ ← $\boxed{\text{very}}$ impt!

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

This is actually why

$$\int_{\vec{a}}^{\vec{b}} \vec{\nabla} f \cdot d\vec{l} = f(\vec{b}) - f(\vec{a})$$

is independent of path!!!

$$= \hat{x}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) + \ldots$$

ϕ

$\boxed{Q}$ Saw this where?

classical mechanics $\vec{F} = -\vec{\nabla} V$

force potential

$W = \int_a^b \vec{F}\cdot d\vec{l}$ is path indep.

(3) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0$ ← also $\boxed{\text{very}}$ impt

(4) $\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \nabla^2\vec{v}$

V14

We saw

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

$$\int_{\vec{a}}^{\vec{b}} \vec{\nabla} f \cdot d\vec{l} = f(\vec{b}) - f(\vec{a})$$

scalar field

Deeper understanding now that this worked because $\vec{\nabla} \times \vec{\nabla} f = 0$!

vector field

volume element

$$\int_V (\vec{\nabla} \cdot \vec{v})\, d\tau = \int_S \vec{v} \cdot d\vec{a}$$

some volume in space

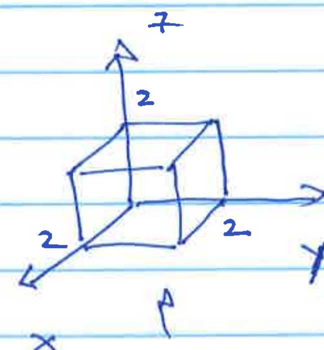
surface enclosing that volume ie boundary

$|d\vec{a}| =$ small surface area; direction is $\perp$ to surface

"Gauss' Law"
"Divergence theorem"

Problem 33 in text

$$\vec{v} = xy\,\hat{x} + 2yz\,\hat{y} + 3xz\,\hat{z}$$



cube with sides of length 2

LHS is simple

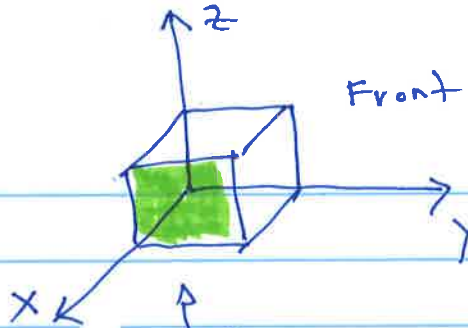
$$\vec{\nabla} \cdot \vec{v} = y + 2z + 3x$$

$\partial v_x/\partial x$ $\quad$ $\partial v_y/\partial y$ $\quad$ $\partial v_x/\partial x$

$$\int_0^2 dx \int_0^2 dy \int_0^2 dz = 4\underbrace{\int_0^2 y\,dy}_{y^2/2|_0^2 = 2} + 8\int_0^2 z\,dz + 12\int_0^2 x\,dx$$

$$= 8 + 16 + 24 = 48.$$

V15



RHS has 6 terms

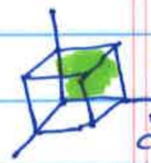
$\vec{v} = xy\,\hat{x} + 2yz\,\hat{y} + 3xz\,\hat{z}$

$\vec{da} = dy\,dz\,\hat{x}$

$\therefore\ \vec{v}\cdot\vec{da} = xy\,dy\,dz$ where $x=2$ on surface

$$\int_S \vec{v}\cdot\vec{da} = \underbrace{\int_0^2\int_0^2}_{\text{both } y \text{ and } z \text{ go from } \phi \text{ to } 2} 2y\,dy\,dz = 8$$

Back



$x=0$

$\vec{da} = -dy\,dz\,\hat{x}$

$$\int_0^2\int_0^2 \underset{x=0}{0}\,y\,dy\,dz = 0$$

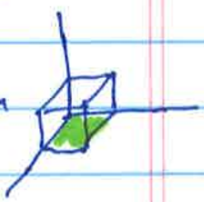
top

$z=2$

$\vec{da} = +dx\,dy\,\hat{z}$

$$\int_0^2\int_0^2 3x\,\underset{z}{2}\,dx\,dy = 24$$

bottom



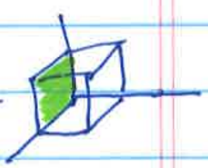
$z=0$

$\vec{da} = -dx\,dy\,\hat{z}$

$= 0$

Sums up to 48!

left

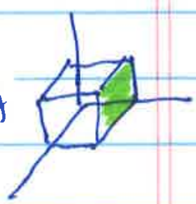


$y=0$

$\vec{da} = -dx\,dz\,\hat{y}$

$= 0$

right



$y=2$

$\vec{da} = +dx\,dz\,\hat{y}$

$$\int_0^2\int_0^2 2\cdot\underset{y}{2}\cdot z\,dx\,dz = 16$$

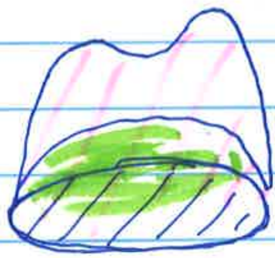
V16

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \int_P \vec{v} \cdot d\vec{l}$$

"Stokes' Theorem"

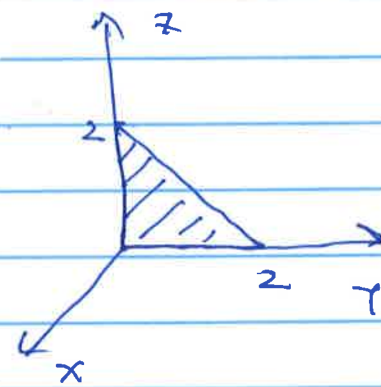
↑ perimeter surrounding S

Note an amazing consequence $\int (\vec{\nabla} \times \vec{v}) \cdot d\vec{a}$ depends only on bdy not on particular surface



all share common boundary  and hence have identical RHS !

Problem 34 in text



$$\vec{v} = xy\,\hat{x} + 2yz\,\hat{y} + 3zx\,\hat{z}$$

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 2yz & 3xz \end{vmatrix}$$

$$= \hat{x}(0 - 2y) - \hat{y}(3z - 0) + \hat{z}(0 - x)$$

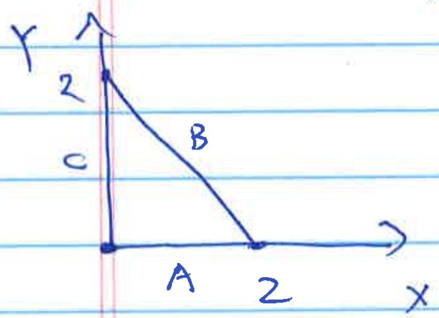
LHS $\quad d\vec{a} = dy\,dz\,\hat{x}$

$$\int_S \vec{\nabla} \times \vec{v} \cdot d\vec{a} = \int_0^2 dy \int_0^{2-y} dz \underbrace{(-2y)}_{\vec{\nabla} \times \vec{v} \cdot d\vec{a}} = -2\int_0^2 (2-y)y\,dy$$

$$= -2\int_0^2 (2y - y^2)\,dy = -2\left(y^2 - y^3/3\right)_0^2 = -8/3$$

V|7

RHS has 3 parts to P $\qquad \vec{v} = xy\,\hat{x} + 2yz\,\hat{y} + 3xz\,\hat{z}$



$\boxed{\text{along A}}$ $\quad \vec{dl} = \hat{x}\,dx \quad$ and $y=0$

$$\int_0^2 \underbrace{dx \quad xy}_{\vec{v}\cdot\vec{dl}} = 0 \quad \text{since } y=0$$

$\boxed{\text{along B}}$ $\quad \vec{dl} = \hat{x}\,dx + \hat{y}\,\underbrace{dy}_{-dx} + \hat{z}\,\underbrace{dz}_{0} \qquad$ and $y = 2-x$, $z=0$

$$\int_2^0 x\overbrace{(2-x)}^{y}\,dx + 2\overbrace{(2-x)}^{y}\,\overbrace{\phi}^{z}\,\overbrace{(-dx)}^{dy}$$

$$\vec{v}\cdot\vec{dl} = \int_2^0 (2x - x^2)\,dx$$

$$= x^2 - \frac{x^3}{3}\Big|_2^0 = -\left(4 - \frac{4}{3}\right)$$

$$= -8/3$$

$\boxed{\text{along C}}$ $\quad \vec{dl} = -dy\,\hat{y}$

$x = 0$
$z = 0$

$$\vec{v}\cdot\vec{dl} = -2yz\,dy$$
$$= 0$$

$$\phi - 8/3 + \phi = -8/3 \qquad \checkmark\checkmark$$