We already looked at $\vec{\nabla}\cdot\vec{v}$ with

$$\vec{v} = \frac{c\,\hat{r}}{r^2} = \frac{c\,\vec{r}}{r^3}$$

and concluded, by working in rectangular coordinates, that $\vec{\nabla}\cdot\vec{v} = 0$ except possibly at origin.

verify in spherical coordinates ← Much easier than calculations in x, y, z

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}1 = 0$$

← except need to remember those funny factors

But look at divergence theorem

$$\int \vec{v}\cdot d\vec{a} = \int c\,R^2\sin\theta\,d\theta\,d\phi\,\frac{c}{R^2}\,\underbrace{\hat{r}\cdot\hat{r}}_{1} = 4\pi c$$

↑ Sphere radius R about origin

$d\vec{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$

$$\underbrace{\int_0^{\pi}\sin\theta\,d\theta}_{2}\;\underbrace{\int_0^{2\pi}d\phi}_{2\pi}$$

But this should equal

$$\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{v}\,d\tau$$

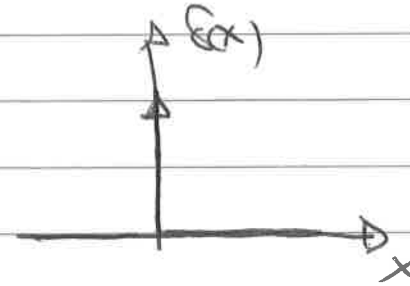
how could this be if $\vec{\nabla}\cdot\vec{v}=0$ everywhere?

$\vec{\nabla}\cdot\vec{v}$ must be nonzero at origin and also so huge that integration over that single point gives nonzero result.

We give a name to the function that is zero everywhere but so big at a single point that you still get a non-zero value when integrating

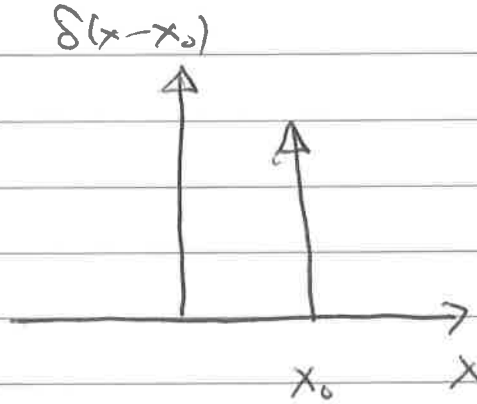
DIRAC DELTA FUNCTION : $\delta(x)$

$$\delta(x) = 0 \quad x \neq 0$$
$$\text{"}\infty\text{"} \quad x = 0$$



$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x) = f(0)$$



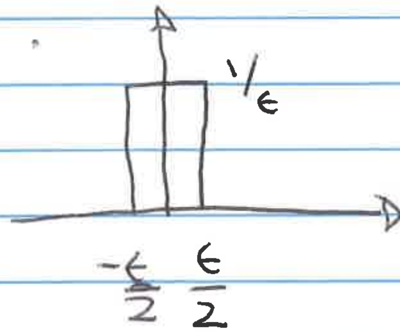
$\delta(x-x_0)$ obviously is nonzero at $x = x_0$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0) = f(x_0)$$

D3.

Think about $\delta(x)$ as limit of family of more conventional functions

$$\delta(x) = \lim_{\epsilon \to 0}$$



Consider $f(x) = 3 - 2x + x^2$

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx \Rightarrow \int_{-\frac{\epsilon}{2}}^{\epsilon/2} f(x)\frac{1}{\epsilon}\,dx$$

$$= \frac{1}{\epsilon}\left(3x - x^2 + \frac{x^3}{3}\right)\Bigg|_{-\epsilon/2}^{\epsilon/2}$$

$$= \frac{1}{\epsilon}\left[3\epsilon - 0 + \frac{\epsilon^3}{12}\right]$$

$$= 3 + \epsilon^2/12 \longrightarrow 3 = f(0)$$

As $\epsilon \to 0$

FTD1

Fourier Transforms and the Dirac Delta Function

You probably know that any function $f(x)$ which is periodic with period $2L$ can be written as

$f(x+2L) = f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$a_n = \frac{1}{L} \int_0^{2L} \cos \frac{n\pi x}{L} f(x)\, dx$$

$$b_n = \frac{1}{L} \int_0^{2L} \sin \frac{n\pi x}{L} f(x)\, dx$$

Using $e^{i\theta} = \cos\theta + i\sin\theta$ this can also be written as

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$c_n = \frac{1}{2L} \int_0^{2L} f(x) e^{-in\pi x/L}$$

FTD2

Proof

$$* \quad c_n = \frac{1}{2L}\int_0^{2L} f(x) e^{-in\pi x/L}\, dx$$

$$= \frac{1}{2L}\int_0^{2L} f(x)\left(\cos\frac{n\pi x}{L} - i\sin\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{2}(a_n - i b_n)$$

Evidently: $c_{-n} = \frac{1}{2}(a_n + i b_n)$

(from $\cos\frac{(-n)\pi x}{L} = \cos\frac{n\pi x}{L}$)

$$c_0 = \frac{1}{2L}\int_0^{2L} f(x)\,dx = \frac{1}{2}a_0$$

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$= c_0 + \sum_1^{\infty} c_n e^{in\pi x/L} + \underbrace{\sum_{-\infty}^{-1} c_n e^{in\pi x/L}}_{\sum_1^{\infty} c_{-n} e^{-in\pi x/L}}$$

$$= c_0 + \sum_1^{\infty} \frac{1}{2}(a_n - ib_n) e^{in\pi x/L} + \frac{1}{2}(a_n + ib_n) e^{-in\pi x/L}$$

$$= \underset{\frac{1}{2}a_0}{\underset{\uparrow}{c_0}} + \sum_1^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

√√

FTD3

Now consider $L \to \infty$ (any function $f(x)$!)

$$k_n = \frac{n\pi}{L} = \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \ldots$$

call this quantity k_n

separated by π/L

continuous as $L \to \infty$

$$dk = \frac{\pi}{L} \overset{1}{dn} \qquad \sum_n \cdot \longrightarrow \int dk$$

$$dn = \frac{L\,dk}{\pi} \qquad f(x) = \int_{-\infty}^{\infty} c(k)\, e^{ikx}\, dk$$

$$\frac{1}{2L} dn = \frac{1}{2\pi} dk \qquad c(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

Much more symmetric than periodic case!

"Fourier Integral"

Suppose $f(x) = \delta(x)$ $\quad c(k) = \frac{1}{2\pi}$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk \qquad !!$$

Reflection of uncertainty relation in QM! If $f(x)$ is perfectly localized $c(k)$ is completely spread out

FTD3A

What is $c(k)$ for a gaussian

$$f(x) = e^{-ax^2} \quad ?$$

HW problem...

Interpret in terms of uncertainty relh.

FTD 4

Application of Fourier integral to classical mechanics

Mass on spring with friction is subjected to external force $f(t)$, Compute $x(t)$.

$x(t)$ obeys Newton's Eqn of motion

$$m\frac{d^2x}{dt^2} = \underbrace{-m\omega_0^2 x}_{\text{spring}} \underbrace{- \gamma\frac{dx}{dt}}_{\text{friction}} \underbrace{+ f(t)}_{\text{external}}$$

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + m\omega_0^2 x = f(t)$$

Fourier expand

$$x(t) = \int_{-\infty}^{\infty} c(\omega)\, e^{i\omega t}\, d\omega$$

Conventional to use ω rather than k in time domain

$$f(t) = \int_{-\infty}^{\infty} b(\omega)\, e^{i\omega t}\, d\omega$$

$$\frac{dx}{dt} = \int_{-\infty}^{\infty} i\omega\, c(\omega)\, e^{i\omega t}\, d\omega$$

$$\frac{d^2x}{dt^2} = \int_{-\infty}^{\infty} -\omega^2 c(\omega)\, e^{i\omega t}\, d\omega$$

PTD 5

$$\int_{-\infty}^{\infty} \left\{ -m\omega^2 + i\gamma\omega + m\omega_0^2 \right\} c(\omega)\, e^{i\omega t}\, d\omega$$

$$= \int_{-\infty}^{\infty} b(\omega)\, e^{i\omega t}\, d\omega$$

$$\left(-m\omega^2 + i\gamma\omega + m\omega_0^2 \right) c(\omega) = b(\omega)$$

$$\therefore \quad x(t) = \int_{-\infty}^{\infty} \frac{b(\omega)}{m(\omega_0^2 - \omega^2) + i\gamma\omega}\, e^{i\omega t}\, d\omega$$

AMAZING! THIS IS FORMULA
FOR $x(t)$ for ARBITRARY $f(t)$
ALL YOU NEED ARE FOURIER COEFFICIENTS

$$f(t) = \int_{-\infty}^{\infty} b(\omega)\, e^{i\omega t}\, d\omega$$