PHYSICS 110A, WINTER 2017
ELECTRICITY AND MAGNETISM

Assignment Eight, Due Monday, March 19, noon.

[1.] A point charge q is located at position $(x, y, z) = (0, 0, d)$. The dielectric constant is ϵ_1 for $z > 0$ and ϵ_2 for $z < 0$. Compute the potential and electric fields at all points in space. Sketch the electric field lines. Hint: The method of images is useful, as in the problem of a point charge near an infinite metallic plane. However, note the boundary conditions are different from that situation, where the electric field must be perpendicular to the surface.

[2.] Griffiths 4-02.

[3.] Griffiths 4-11.

[4.] Griffiths 4-18.

[5.] Griffiths 4-26.

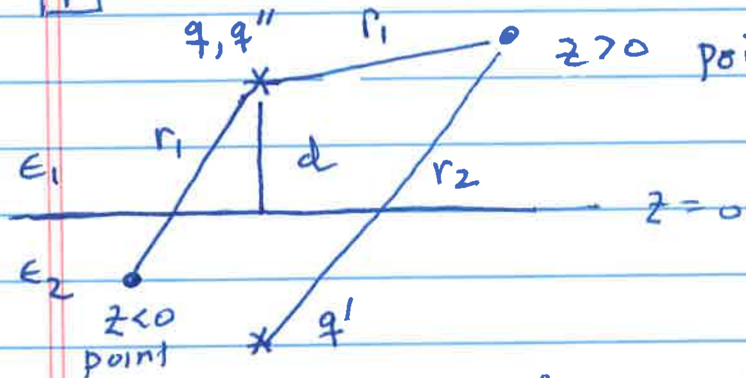
[6.] Griffiths 4-31.

[7.] Extra credit: A sphere of dielectric constant ϵ is placed in a uniform external electric field. Compute the potential everywhere in space, as well as the volume(surface) polarization charges within(on) the sphere.

1-1

P110A Assignment 8

$\boxed{1}$



We guess an "image method solution for which at z>0

$$V = \frac{1}{4\pi\epsilon_1}\left(\frac{q}{r_1} + \frac{q'}{r_2}\right) \qquad z>0$$

$$V = \frac{1}{4\pi\epsilon_2}\left(\frac{q''}{r_1}\right) \qquad z<0$$

This choice is guided by the principle that we cannot alter the charge in the region of space under consideration. Thus for $z>0$ we must have a charge q at $(0,0,d)$ and nowhere else in $z>0$ and for $z<0$ we must have no charge anywhere in the region $z<0$. We are, however, free to put (or to alter) charges outside the region being considered

Our challenge is to pick q, q', q'' to match boundary conditions.

1-2.

$$\vec{E} = -\nabla V$$

We know E_t is continuous, so E_x and E_y are the same on the ϵ_1 and ϵ_2 sides of the interface

ϵ_1 side: $$-\frac{\partial}{\partial x}\frac{1}{4\pi\epsilon_1}\left\{\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}+\frac{q'}{\sqrt{x^2+y^2+(z+d)^2}}\right\}\Bigg|_{z=0}$$

ϵ_2 side $$= -\frac{\partial}{\partial x}\frac{1}{4\pi\epsilon_2}\left\{\frac{q''}{\sqrt{x^2+y^2+(z-d)^2}}\right\}\Bigg|_{z=0}$$

$$\Rightarrow \frac{1}{\epsilon_1}(q+q') = \frac{1}{\epsilon_2}q''$$

Meanwhile, since there is no free charge on the surface $D_n = D_z$ is continuous. Recall $\vec{D} = \epsilon\vec{E}+\vec{P}$

ϵ_1 side $$-\epsilon_1\frac{\partial}{\partial z}\frac{1}{4\pi\epsilon_1}\left\{\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}+\frac{q'}{\sqrt{x^2+y^2+(z+d)^2}}\right\}\Bigg|_{z=0}$$

ϵ_2 side $$= -\epsilon_2\frac{\partial}{\partial z}\frac{1}{4\pi\epsilon_2}\left\{\frac{q''}{\sqrt{x^2+y^2+(z+d)^2}}\right\}\Bigg|_{z=0}$$

$$\Rightarrow \quad q - q' = q''$$

NB. I used $$\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+(z\pm d)^2}}\Bigg|_{z=0} = \frac{-x}{\sqrt{x^2+y^2+(z\pm d)^2}}\Bigg|_{z=0}$$

$$= {-x}/\sqrt{x^2+y^2+d^2}$$

and $$\frac{\partial}{\partial z}\frac{1}{\sqrt{x^2+y^2+(z\pm d)^2}}\Bigg|_{z=0} = \frac{-(z\pm d)}{\sqrt{x^2+y^2+(z\pm d)^2}}\Bigg|_{z=0} = \frac{\mp d}{\sqrt{x^2+y^2+d^2}}$$

1-3

We can solve for $q' = \left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right) q$

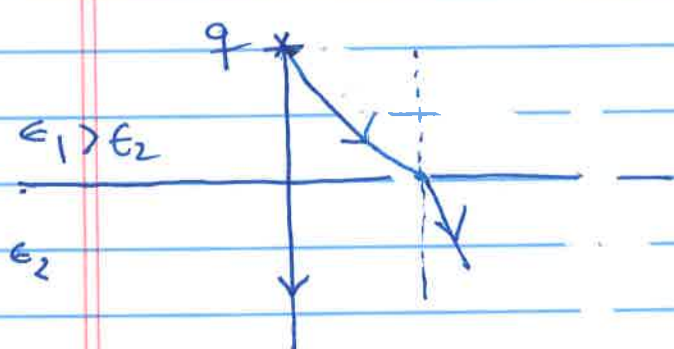
$$q'' = \left(\frac{2\epsilon_2}{\epsilon_1 + \epsilon_2}\right) q$$

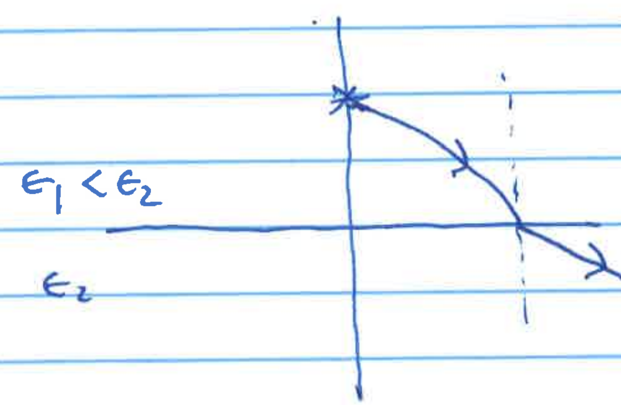
These formulae make sense in the limit $\epsilon_1 = \epsilon_2$ since then there is no interface and $q' = 0$ and $q'' = q$

We have $D_{1n} = D_{2n} \Rightarrow \epsilon_1 E_{1n} = \epsilon_2 E_{2n}$

If $\epsilon_1 > \epsilon_2$ $\quad E_{2n} = \frac{\epsilon_1}{\epsilon_2} E_{1n} \Rightarrow$ the normal component of E_2 is bigger than the normal component of E_1.

$\Rightarrow$ E_2 bends towards normal.





$\boxed{2}$ Griffiths 4-2

This is a mathematically slightly more complex version of the problem in class, where we modeled the e^- cloud as a uniform ball of charge. But the idea is the same: The external field shifts the position of the e^- ball until its force on the proton is identical to the field's, leading to equilibrium.

To find the field here use $\Phi_E = 1/\epsilon_0 \, Q_{enc}$

$$E \, 4\pi r^2 = \frac{1}{\epsilon_0} \int_0^r 4\pi r'^2 dr' \; q/\pi a^3 \, e^{-2r'/a}$$

$$= \frac{4q}{a^3 \epsilon_0} \left\{ \int_0^r \underset{u}{r'^2} \, \underbrace{e^{-2r'/a} dr'}_{dv} \right\}$$

$$r'^2 \left(-\frac{a}{2}\right) e^{-2r'/a} \Big|_0^r + a \int_0^r \underset{u}{r'} \, \underbrace{e^{-2r'/a} dr'}_{dv}$$

$$-\frac{ar^2}{2} e^{-2r/a} + a \left\{ r' \left(-\frac{a}{2}\right) e^{-2r'/a} \Big|_0^r + a/2 \int_0^r e^{-2r'/a} dr' \right\}$$

$$= -\frac{ar^2}{2} e^{-2r/a} - \frac{a^2 r}{2} e^{-2r/a} - \frac{a^3}{4} \left(e^{-2r/a} - 1\right)$$

$$E = \frac{q}{4\pi r^2 \epsilon_0} \left\{ 1 - e^{-2r/a} \left(1 + 2r/a + 2r^2/a^2 \right) \right\}$$

Check: @ $r \to \infty$ $E = q/4\pi r^2 \epsilon_0$ ✓✓

@ $r \to 0$ … see next page

2-2

Assuming r is small (e^- cloud not shifted much)

$$E = \frac{q}{4\pi r^2 \epsilon_0}\left\{1 - \left(1 - \frac{2r}{a} + \frac{1}{2}\left(\frac{2r}{a}\right)^2 - \frac{1}{6}\left(\frac{2r}{a}\right)^3 + \dots\right)\right.$$

$$\left. \cdot\left(1 + {}^{2r}/_{a} + {}^{2r^2}/_{a^2}\right)\right\}$$

$$= \frac{q}{4\pi\epsilon_0 r^2}\left\{1 - 1 - \frac{2r}{a} - \frac{2r^2}{a^2}\right.$$

$$+ \frac{2r}{a} + \frac{4r^2}{a^2} + \frac{4r^3}{a^3}$$

$$- \frac{2r^3}{a^2} - \frac{4r^3}{a^3} - \dots$$

$$\left. + \frac{4}{3}\frac{r^3}{a^3} - \dots \right\}$$

$$= \frac{q}{4\pi\epsilon_0 r^2}\,\frac{4}{3}\,{}^{r^3}/_{a^3} = \frac{q r}{3\pi\epsilon_0 a^3}$$

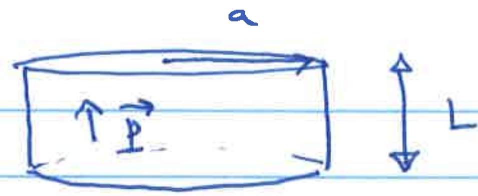
This field must equal the external one applied, so the dipole moment induced, $qr \Rightarrow p$, is related to E by

$$E = \underbrace{\frac{1}{3\pi\epsilon_0 a^3}}_{1/\alpha}\, p \quad \swarrow qr$$

$$\alpha = 3\pi\epsilon_0 a^3$$

3-1

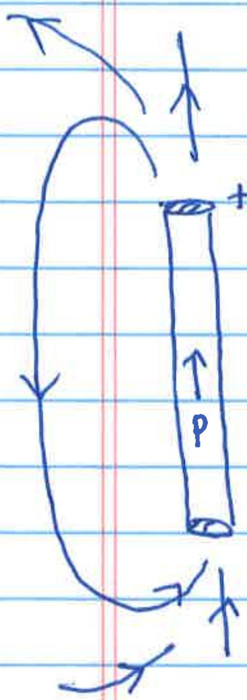
$\boxed{3}$ Griffiths 4-11



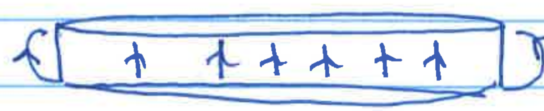
Since $\vec{P}$ = const $\quad \rho_b = -\vec{\nabla}\cdot\vec{P} = 0$

$\sigma_b = \vec{P}\cdot\hat{n} = +P$ at one end

$-P$ at the other



like 2 point charges (dipole)

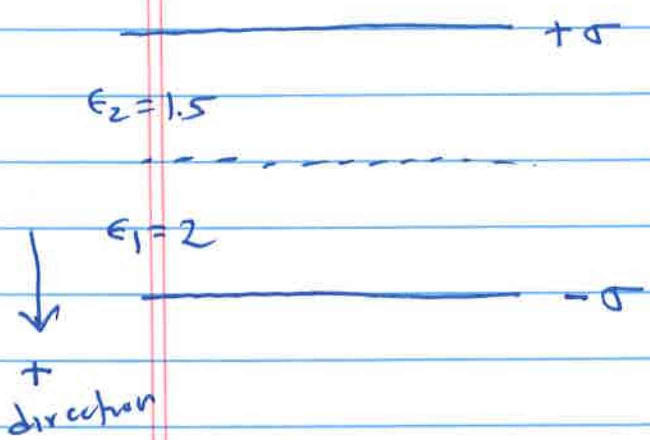


like capacitor

"fringing fields"

4-1

$\boxed{4}$ Griffiths 4-18



a) $D = \sigma$ everywhere

since D only responds to free charge

b) $E = \sigma/\epsilon_1$ in slab 1

$E = \sigma/\epsilon_2$ in slab 2

$(E = D/\epsilon)$

c) $P = D - \epsilon_0 E = \sigma - \epsilon_0 \sigma/2\epsilon_0 = \sigma/2$ ← slab # 1

$= \sigma - \epsilon_0 \sigma/1.5\epsilon_0 = \dfrac{\sigma}{3}$ ← slab # 2

d) $V = \int E \cdot dl = \dfrac{\sigma}{2\epsilon_0} a + \dfrac{\sigma}{1.5\epsilon_0} a = \dfrac{7\sigma a}{6\epsilon_0}$

e) $\sigma = \vec{P} \cdot \hat{n}$

bottom of slab #1 $\sigma_b = \sigma/2$

top of slab #1 $\sigma_b = -\sigma/2$

recall $P_1 = \dfrac{\sigma}{2\epsilon_0}$

$(\sigma_b)_{tot} = -\sigma/6$

bottom of slab #2 $\sigma_b = +\sigma/3$

top of slab #2 $\sigma_b = -\sigma/3$

$P_2 = \dfrac{\sigma}{3\epsilon_0}$

f) $$E_1 = \frac{1}{2}\Big\{ \underbrace{\frac{\sigma}{\epsilon_0}}_{\text{free charge at top of slab 2}} - \underbrace{\frac{\sigma}{3\epsilon_0}}_{\text{bound at top of slab 2}} + \underbrace{\frac{\sigma}{3\epsilon_0}}_{\text{bound bottom slab 2}} - \underbrace{\frac{\sigma}{2\epsilon_0}}_{\text{bound top slab 1}} - \underbrace{\frac{\sigma}{2\epsilon_0}}_{\text{bound bottom slab 1}} + \underbrace{\frac{\sigma}{\epsilon_0}}_{\text{free bottom slab 1}} \Big\} = \frac{\sigma}{2\epsilon_0}$$

$\boxed{5}$ Griffiths 4-26

We did not cover energy of dielectrics in class, but reading the text

$$\text{Energy} = \frac{1}{2}\int \vec{D}\cdot\vec{E}\,d\tau$$

we get

$$D = \begin{cases} 0 & r<a \\ Q/4\pi r^2 & r>a \end{cases}$$

$\Leftarrow$ D responds only to free charge Q on surface of conductor

$$E = \begin{cases} 0 & r<a \\ Q/4\pi\epsilon r^2 & a<r<b \\ Q/4\pi\epsilon_0 r^2 & b<r \end{cases}$$

$\Leftarrow$ E reduced by ϵ

$$\text{Energy} = \int_0^a 0\,d\tau + \int_a^b \frac{1}{2}\frac{Q}{4\pi r^2}\frac{Q}{4\pi\epsilon r^2}4\pi r^2 dr$$

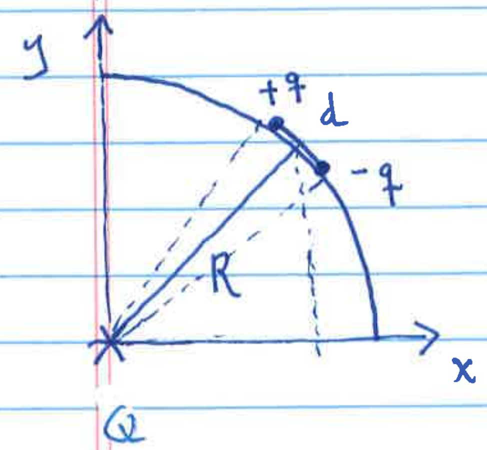
$$+ \int_b^\infty \frac{1}{2}\frac{Q}{4\pi r^2}\frac{Q}{4\pi\epsilon_0 r^2}4\pi r^2 dr \Big\}$$

$$= \frac{Q^2}{8\pi\epsilon}\left.\frac{-1}{r}\right|_a^b + \frac{Q^2}{8\pi\epsilon_0}\left.\frac{-1}{r}\right|_b^\infty$$

$$= \frac{Q^2}{8\pi}\left\{\frac{1}{\epsilon}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{\epsilon_0 b}\right\}$$

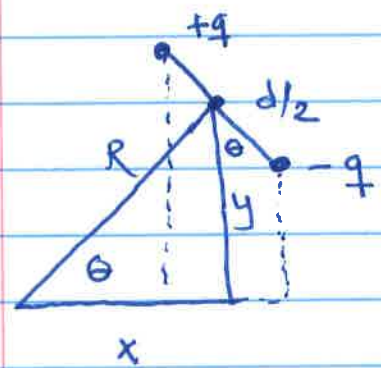
[6] Griffiths 4-31

Probably best to use polar coordinates, but let's be primitive and use rectangular ones!



Position of dipole center is at (x, y) with $x^2 + y^2 = R^2$

Position of $-q$ charge is



$$X_- = x + \frac{d}{2}\sin\theta = x + \frac{d}{2}\, y/R$$

$$Y_- = y - \frac{d}{2}\cos\theta = y - \frac{d}{2}\frac{x}{R}$$

likewise $+q$ is located at

$$(X_+, Y_+) = \left(x - dy/2R,\ y + xd/2R\right)$$

Can compute forces on each one

$$F_{x,-} = \frac{Qq}{4\pi\epsilon_0 r_-^2}\,\frac{X_-}{r_-}$$

$$F_{x,tot} = \frac{Qq}{4\pi\epsilon_0}\left(-\frac{X_-}{r_-^3} + \frac{X_+}{r_+^3}\right) = \frac{Qq}{4\pi\epsilon_0}\left(-\frac{x + dy/2R}{r_-^3} + \frac{x - \frac{dy}{2R}}{r_+^3}\right)$$

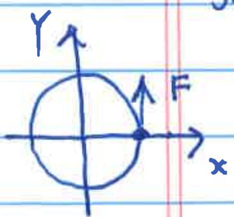
To lowest order we can set $r_+ = r_- = R$

$$F_{x,tot} = -\frac{Qq}{4\pi\epsilon_0 R^3}\,\frac{dy}{R}$$

6-2

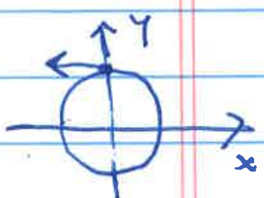
An identical calculation $F_{y\,tot} = \dfrac{Qq}{4\pi\epsilon_0 R^3}\,\dfrac{dx}{R}$

So, for example, if dipole is at $(X,Y) = (R,0)$



$$F_{y\,tot} = \frac{Qqd}{4\pi\epsilon_0 R^3} \qquad F_{x\,tot} = 0$$

and if dipole is at $(x,y) = (0,R)$



$$F_{y\,tot} = 0 \qquad F_{x\,tot} = \frac{-Qqd}{4\pi\epsilon_0 R^3}$$

Force is always forward!

The perpetual motion machine fallacy is that we are neglecting the forces which keep dipole oriented tangential to track.

$\boxed{7}$ Griffiths does this problem. It is Example #7

of chapter 4 , pages 193-194.