# PHYSICS 110A, WINTER 2017
# ELECTRICITY AND MAGNETISM

## Assignment Seven, Due Friday, March 9, 5:00 pm.

**[1.]** Griffiths 3-24.

**[2.]** An infinite cylinder of radius $r = a$ has charge density $\sigma(\phi) = \sigma_0 \cos 4\phi$ on its surface. Compute the potential $V(\rho, \phi, z)$ inside (i.e. for $\rho < a$) and also outside ($\rho > a$).

**[3.]** Extra credit: Go through separation of variables solution of Laplace's equation in cylindrical coordinates when it is *not* legitimate to ignore the $z$ dependence. (This is an extension of Problem [1].) Determine what the functional form is for the $z$ and $\phi$ dependence. Look up online what the solutions are for the $\rho$ dependence, just so you know in the future what those functions are.

**[4.]** Compute the monopole, dipole, and quadrupole terms of the potential of a point charge $q$ located at position $\mathbf{r} = (0, 0, a)$. What is true for the case $a = 0$?

**[5.]** Compute the monopole, dipole, and quadrupole terms of the potential of two point charges, $+q$ located at position $\mathbf{r} = (0, 0, a)$, and $-q$ located at position $\mathbf{r} = (0, 0, b)$. Comment on the dependence of the dipole term on $a, b$. In particular, if you shift the charges by the same amount in the $z$ direction, to $(0, 0, a + c)$ and $(0, 0, b + c)$, what happens to the dipole term? Enunciate a general rule about the monopole and dipole terms, the nature of the charge distribution, and the choice of origin.

Physics 110A Winter 2018
Assignment 7

$\boxed{1}$ See the solution to problem #2, where

we conclude solns are of the form

$$\rho^n \cos n\phi;\ \rho^n \sin n\phi;\ \rho^{-n} \cos n\phi,\ \rho^{-n} \sin n\phi$$

where $n = 1, 2, 3, \ldots$ is a positive integer.

To this list we must also add the case where $n = 0$
so there is no $\phi$ dependence and

$$\rho\, \partial/\partial\rho\ \rho\, \partial/\partial\rho\ R(\rho) = \phi$$

This has the soln $R(\rho) = C$, a constant, but also $R(\rho) = \ln\rho$
since

$$\rho\, \partial/\partial\rho\ \ln\rho = \rho\, 1/\rho = 1$$

So that when we take the second $\rho\,\partial/\partial\rho$ in the Laplacian
we get zero. This $\ln\rho$ term is of course the potential
due to an infinite line of charge.

$\boxed{2}$ In cylindrical coordinates $(\rho, \phi, z)$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial V}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$$

If there is no $z$ dependence, and we write $V(\rho,\phi) = R(\rho)\,Q(\phi)$ then Laplace's eqn $\nabla^2 V = 0$ becomes

$$Q\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R + \frac{1}{\rho^2}R\frac{\partial^2 Q}{\partial\phi^2} = 0$$

$$\Rightarrow \quad \frac{1}{R}\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R = -\frac{1}{Q}\frac{\partial^2 Q}{\partial\phi^2} = k^2$$

Clearly $Q(\phi) = \sin k\phi\,;\ \cos k\phi \leftarrow$ Since $V$ has period $2\pi$ $k$ must be $n$, an integer

If we guess that $R(\rho) = \rho^l$ then

$$\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R = \rho\frac{\partial}{\partial\rho}\rho\, l\rho^{l-1} = \rho l^2\rho^{l-1} = l^2 R$$

So this works if $l = \pm k = \pm n$

We conclude the solutions of $\nabla^2 V = 0$ have the form

$$V(\rho,\phi) = \begin{array}{l} \rho^n \sin n\phi\,;\ \rho^n \cos n\phi \\ \rho^{-n}\sin n\phi\,;\ \rho^{-n}\cos n\phi \end{array}$$

Clearly the top line are the ones to use for $\rho < a$ and the bottom line for $\rho > a$, to avoid divergences at $\rho = 0$ and $\rho = \infty$

2-2

$$\rho < a \qquad V(\rho,\phi) = \sum_{n=1}^{\infty} \rho^n (a_n \cos n\phi + b_n \sin n\phi)$$

$$\rho > a \qquad V(\rho,\phi) = \sum_{n=1}^{\infty} \rho^{-n} (c_n \cos n\phi + d_n \sin n\phi)$$

We want $V$ continuous at $\rho = a$ and also that the discontinuity in $\frac{\partial V}{\partial \rho}$, the normal piece of $\vec{E}$ be $\sigma/\epsilon_0 = \frac{1}{\epsilon_0}\sigma_0 \cos 4\phi$.

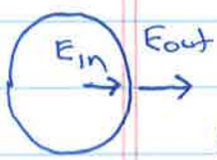
This suggests we try a solution with $b_n = d_n = 0$ and only $a_4$ and $c_4$ nonzero

$$\rho < a \qquad V(\rho,\phi) = \rho^4 a_4 \cos 4\phi$$

$$\rho > a \qquad V(\rho,\phi) = \rho^{-4} c_4 \cos 4\phi$$

Continuity of $V(a,\phi) \Rightarrow a^4 a_4 = a^{-4} c_4 \Rightarrow c_4 = a^8 a_4$

Discontinuity in $\underset{\displaystyle -\frac{\partial V}{\partial \rho}}{\overset{\shortparallel}{E_n}} \Rightarrow \left. +4\rho^{-5} c_4 \cos 4\phi + 4\rho^3 a_4 \cos 4\phi \right|_{\rho=a} = \frac{1}{\epsilon_0}\sigma_0 \cos 4\phi$




Cancelling the $\cos 4\phi$ factors, and using $c_4 = a^8 a_4$

$\underbrace{E_{out} A - E_{in} A}_{\phi_E} = \sigma/\epsilon_0 A$

$$4 a^{-5} a^8 a_4 + 4 a^3 a_4 = \sigma_0/\epsilon_0$$

$E_{out} - E_{in} = \sigma/\epsilon_0$

$$a_4 = \sigma_0 / 8\epsilon_0 a^3$$

$$V(\rho,\phi) = \begin{cases} \sigma_0/8\epsilon_0 a^3 \, \rho^4 \cos 4\phi & \rho < a \\ \sigma_0/8\epsilon_0 a^3 \, \rho^{-4} \cos 4\phi & \rho > a \end{cases}$$

3-1

Extra credit.

$\boxed{3.}$ Same starting point as problem 2, except

$$V(\rho, \phi, z) = R(\rho) Q(\phi) T(z)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\, r\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \Leftarrow$$ Using Laplacian in cylindrical coordinates

$$QT\frac{d^2R}{dr^2} + \frac{QT}{r}\frac{dR}{dr} + \frac{RT}{r^2}\frac{d^2Q}{d\phi^2} + RQ\frac{d^2T}{dz^2} = 0$$

divide by RQT

$$\underbrace{\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{r^2 Q}\frac{d^2Q}{d\phi^2}}_{r,\phi \text{ dependence only}} = \underbrace{-\frac{1}{T}\frac{d^2T}{dz^2}}_{z \text{ dependence only}}$$

$\therefore$ = constant. Let's call it $-\ell^2$

$$T(z) = a e^{\ell z} + b e^{-\ell z} \Leftarrow \frac{d^2T}{dz^2} = \ell^2 T$$

(or $\cosh \ell z$, $\sinh \ell z$)

$$\underbrace{\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} + \ell^2 r^2}_{r \text{ only}} = \underbrace{-\frac{1}{Q}\frac{d^2Q}{d\phi^2}}_{\phi \text{ only}}$$

$\therefore$ = constant. Let's call it $m^2$

$$Q(\phi) = c e^{im\phi} + d e^{-im\phi} \Leftarrow \frac{d^2Q}{d\phi^2} = -m^2 Q$$

(or $\sin m\phi$; $\cos m\phi$)

Note: $m$ must be an integer since $V(\phi + 2\pi) = V(\phi)$

The r dependence is ugly!

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\ell^2 r^2 - m^2) = 0$$

Define $x = \ell r$

$$x^2 \frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - m^2) = 0$$

This is the Bessel Eqn

You can stop here.. But I cite a few items

m noninteger

$$\begin{cases} J_m(x) = x^m \sum_0^\infty (-1)^n x^{2n} / 2^{m+2n} n!\, \Gamma(m+n+1) \\ J_{-m}(x) = x^{-m} \sum_0^\infty (-1)^n x^{2n} / 2^{-m+2n} n!\, \Gamma(-m+n+1) \end{cases}$$

$$R(x) = e J_m(x) + f J_{-m}(x) \qquad \Gamma(v) \equiv \int_0^\infty e^{-t} t^{v-1} dt$$

m integer $J_{-m}(x) = (-1)^m J_m(x)$ and instead second soln to Bessel eqn is

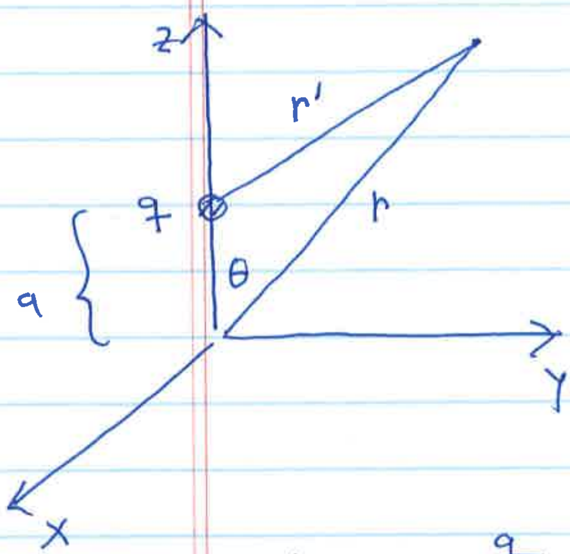
$$Y_m(x) = \lim_{p \to m} \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi}$$

$$R(x) = e J_m(x) + f Y_m(x)$$

See posted notes for extensive discussion and applications: Diffusion, Schroedinger, Stat Mech!

4-1

$\boxed{4}$ We did this in class...




$$V = \frac{1}{4\pi\epsilon_0} q/r'$$

$$= \frac{1}{4\pi\epsilon_0} q/(r^2 - 2ra\cos\theta + a^2)^{1/2}$$

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0 r}\left\{1 - \frac{1}{2}\left(-2\frac{a\cos\theta}{r} + \frac{a^2}{r^2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-2\frac{a\cos\theta}{r} + \frac{a^2}{r^2}\right)^2 + \dots\right\}$$

$$= \frac{q}{4\pi\epsilon_0 r}\left\{1 + \frac{a}{r}\cos\theta + \frac{a^2}{r^2}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \dots\right\}$$

$P_0(\cos\theta)$ — monopole — $q/4\pi\epsilon_0 r$

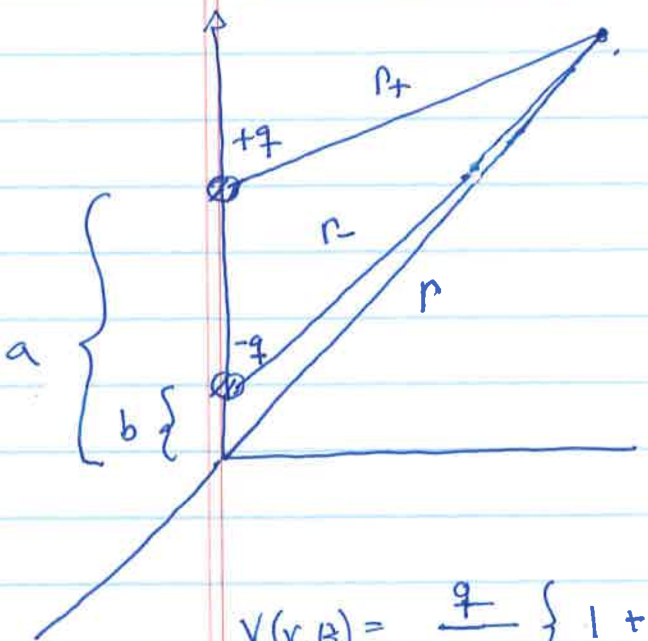
$P_1(\cos\theta)$ — dipole — $aq/4\pi\epsilon_0 r^2 \cos\theta$

$P_2(\cos\theta)$ — quadrupole — $qa^2/4\pi\epsilon_0 r^3\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$

If $a = 0$ only the monopole term survives.

5-1

$\boxed{5}$




$$V(r,\theta) = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{r_+} - \frac{1}{r_-}\right\}$$

$$= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{(r^2 - 2ra\cos\theta + a^2)^{1/2}} - \frac{1}{(r^2 - 2rb\cos\theta + b^2)^{\frac{1}{2}}}\right\}$$

Following #4 we see:

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0}\left\{1 + \frac{a}{r}\cos\theta + \frac{a^2}{r^2}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \cdots \right.$$
$$\left. -1 - \frac{b}{r}\cos\theta - \frac{b^2}{r^2}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \cdots \right\}$$

* There is no monopole term if $q_{tot} = +q - q = 0$.

* The dipole term is $\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}(a-b)\cos\theta$

and depends only on the $+q$ / $-q$ separation.

It is unaffected by shifting the charges by a common amount. Put alternately, the dipole term is invariant under changes of the origin.