PHYSICS 110A, WINTER 2017
ELECTRICITY AND MAGNETISM

Assignment Five, Due Friday, February 16, 5:00 pm.

[1.] Griffiths Problem 7, Chapter 3.

[2.] Griffiths Problem 12, Chapter 3.

[3.] Find the analytic solution of the one-dimensional Poisson equation with $\rho(x) = 12\,\epsilon_0\,x^2$,

$$-\frac{d^2\phi}{dx^2} = 12\,x^2,$$

with boundary conditions $\phi(0) = \phi(1) = 0$. What is the value of the quantity,

$$E = \int_0^1 [\,\frac{1}{2}(\frac{d\phi}{dx})^2 - 12x^2\phi\,]dx\ ?$$

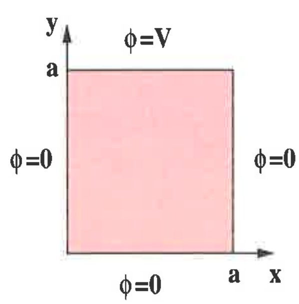
and what is its physical significance?

[4.] (extra credit) Solve problem 3 numerically. Use $dx = 0.1$. Make a plot containing $\phi(x)$ at a few times during the course of the iteration, and also containing the analytic solution. (I will talk about this problem on Wednesday in the "problem session" if you need guidance.)

[5.] (extra, extra credit) Solve Laplace's equation $\nabla^2\phi = 0$ for the potential $\phi(x,y)$ for the square region in the figure, with the boundary conditions shown. Use the iterative method discussed in class, whose form in $d = 2$ (with $\rho = 0$) is,

$$\phi(i,j) = \frac{1}{4}[\phi(i+1,j) + \phi(i-1,j) + \phi(i,j-1) + \phi(i,j+1)]\ .$$

Use $dx = 0.1$ and $a = 1$. Compare your solution with the analytic one obtained in class by plotting the two results for $\phi(x, y = a/2)$, $0 < x < a$ on the same graph. Make a second graph comparing the analytic and numeric solutions for a vertical cut, $\phi(x = a/2, y)$, $0 < y < a$.

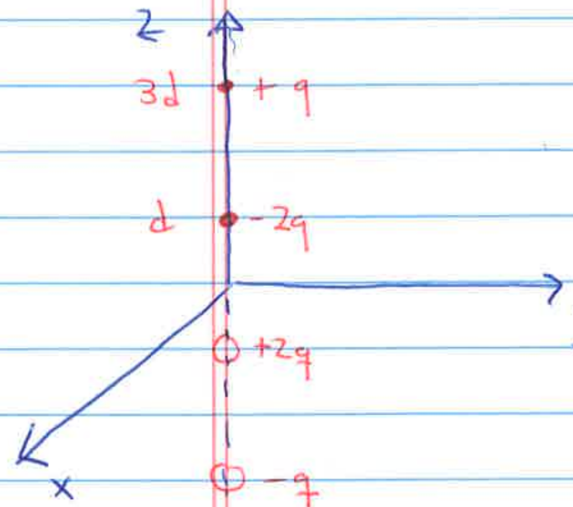


1.

Physics 110A PS5 Sol'ns

1 Griffiths 3-7

Clearly there must be image charges below the plane as shown here: Then the force on $+q$ is



$$\vec{F} = \frac{1}{4\pi\epsilon_0} q^2 \hat{z} \left\{ \frac{-2}{(2d)^2} + \frac{2}{(4d)^2} - \frac{1}{(6d)^2} \right\}$$

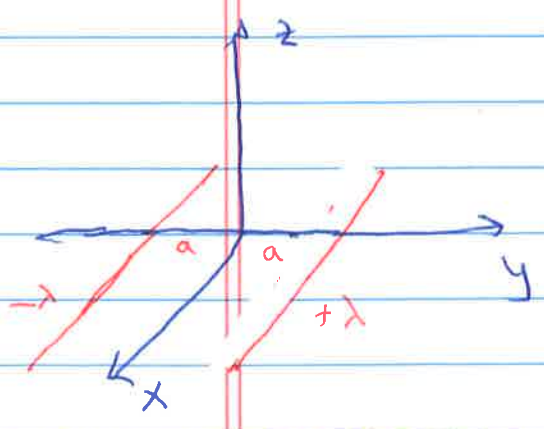
$$= \frac{q^2 \hat{z}}{4\pi\epsilon_0 d^2} \left\{ -\frac{1}{2} + \frac{1}{8} - \frac{1}{36} \right\}$$

$$\frac{1}{72}\left\{-36 + 9 - 2\right\}$$

$$\left(-29/72\right)$$

2 Griffiths 3-12

As mentioned in my email, the first step here is to do problem 2.52. Let's do it! We are asked for the potential of two long wires running along $\hat{x}$ axis



We know from Gauss' law

$$E 2\pi r L = \frac{\lambda L}{\epsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\epsilon_0 r}$$

is field of long straight wire

2.

Hence the potential is

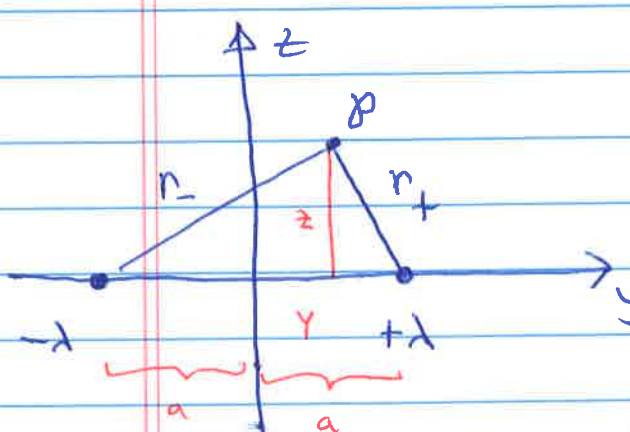
$$V(r) = -\int_{r_0}^{r} E(r')\,dr' = \frac{-\lambda}{2\pi\epsilon_0} \ln r/r_0$$

some origin

For a pair of wires with $\pm\lambda$

at P: $$V(r) = -\frac{\lambda}{2\pi\epsilon_0}\left(\ln\frac{r_+}{r_0} - \ln\frac{r_-}{r_0}\right)$$

$$= \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_-}{r_+}\right)$$ origin r_0 goes away!



$$V(y,z) = \frac{\lambda}{4\pi\epsilon_0}\ln\frac{(y+a)^2+z^2}{(y-a)^2+z^2}$$

If $V = \text{const} = V_0$ we need

$$\frac{(y+a)^2+z^2}{(y-a)^2+z^2} = e^{4\pi\epsilon_0 V_0/\lambda} = b$$

$\begin{cases} b>1 \text{ if } V_0>0 \\ b<1 \text{ if } V_0<0 \end{cases}$

$$(y+a)^2+z^2 = b(y-a)^2+z^2 b$$

$$y^2+2ay+a^2+z^2 = by^2-2aby+ba^2+bz^2$$

$$(1-b)y^2+2a(1+b)y+(1-b)z^2+(1-b)a^2=0$$

3. Complete the square

$$(1-b)\left[y + \frac{a(1+b)}{1-b}\right]^2 - \frac{a^2(1+b)^2}{(1-b)}$$

$$+ (1-b)z^2 + (1-b)a^2 = 0$$

$$\left[y + \frac{a(1+b)}{(1-b)}\right]^2 + z^2 = \frac{1}{1-b}\left\{\frac{a^2(1+b)^2}{1-b} - (1-b)a^2\right\}$$

$\Rightarrow$ circle with center at (y_0, z_0)

$$y_0 = a(1+b)/(1-b) \quad \leftarrow \text{ note that } a^2 + y_0^2$$

$$= a^2\left[1 + \frac{(1+b)^2}{(1-b)^2}\right]$$

$$z_0 = 0$$

$$= a^2\, 4b/(1-b)^2$$

The radius is square root of rhs $\qquad \leftrightarrow R^2$

$$R^2 = \frac{a^2}{(1-b)^2}\underbrace{\left\{(1+b)^2 - (1-b)^2\right\}}_{4b} = \frac{4a^2 b}{(1-b)^2}$$

$$R = 2a\sqrt{b}/(1-b)$$

These eqns tell us equipotentials are cylinders with axes along $\hat{x}$, but notice center $y_0 \neq a$! The cylinders are $\underbrace{\text{not}}$ centered on the lines of charge.

(This is a bit counterintuitive at first, but not if you think carefully!)

4.

We now find 3-12 Easy.

$$y_0 = \frac{a(1+b)}{(1-b)} \qquad R = \frac{2a\sqrt{b}}{(1-b)} \qquad b = e^{4\pi\epsilon_0 V/\lambda}$$

Eqn for position and radius of equipotential in terms of position "a" of lines and potential V_0

Translate y, z of 2-52 into x, y of 3-11

$$V = \frac{\lambda}{4\pi\epsilon_0} \ln \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}$$

The location of the centers of the equipotential is called d in 3-11 and is y_0 in 2-52

$a(1+b)/_{1-b} = d$

We need to figure out what λ and a are in terms of the given R, d, V_0

In problem 2-52 we saw $a^2 + y_0^2 = R^2$

So we immediately get $a^2 = R^2 - d^2 \quad a = \sqrt{R^2 - d^2}$

The other condition is

$$\frac{y_0}{R} \to \frac{d}{R} = \frac{a(1+b)/(1-b)}{2a\sqrt{b}/(1-b)} = \frac{1}{2}\frac{1+b}{\sqrt{b}} = \frac{1}{2}\left(\sqrt{b} + \frac{1}{\sqrt{b}}\right)$$

But $b = e^{4\pi\epsilon_0 V_0/\lambda}$ so $\sqrt{b} + \frac{1}{\sqrt{b}} = 2\cosh\frac{2\pi\epsilon_0 V_0}{\lambda}$

So given d, R, V_0 we get λ from

$$d/R = 2\cosh\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right) \qquad \lambda = \frac{2\pi\epsilon_0 V_0}{\cosh^{-1}(d/R)}$$

5,

$\boxed{3}$ $-d^2\phi/dx^2 = +12x^2$

$d\phi/dx = A - 4x^3$ $\phi(0) = B = 0$

$\phi(1) = A - 1 = 0 \quad A = 1$

$\phi = Ax - x^4 + B$

$\phi(x) = x - x^4$

$$E = \int_0^1 \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - 12x^2\phi\right] dx$$

$$= \int_0^1 \left[\frac{1}{2}(1-4x^3)^2 - 12x^2(x-x^4)\right] dx$$

$$= \int_0^1 \left(\frac{1}{2} - 4x^3 + 8x^6 - 12x^3 + 12x^6\right) dx$$

$$= \int_0^1 \left(1/2 - 16x^3 + 20x^6\right) dx = \frac{1}{2}x - 4x^4 + \frac{20}{7}x^7 \Big|_0^1$$

$$= 1/2 - 4 + 20/7 = 1/14\{7 - 56 + 40\} = -9/14.$$

This looks just like a classical mechanics problem (calculus of variations) where we are asked to minimize

$$S = \int \left\{1/2\left(\frac{dx}{dt}\right)^2 - U(x)\right\} dx = \int \mathcal{L}\, dx$$

The Lagrange eqn is $d/dt\left(\partial\mathcal{L}/\partial\dot{x}\right) - \frac{\partial\mathcal{L}}{\partial x} = 0$

In our case, translating notation: $\frac{d}{dx}\frac{\partial\phi}{\partial x} - (-12x^2) = 0$

$x \to \phi \quad t \to x$

$d^2\phi/dx^2 = -12x^2$

So: physical interpretation: Sol'n to Laplace Eqn minimizes E.

```
      implicit none
      integer it,Nit,i,N
      real*8 h,phi(0:1000),s(0:1000),onemw,h2,w
      real*8 phinew(0:1000),E,wo2,x

      write (6,*) 'enter N,h'
      read  (5,*)         N,h
      write (67,*) 'N,h'
      write (67,*)  N,h
      write (6,*) 'enter Nit,w'
      read  (5,*)         Nit,w
      write (67,*) 'Nit,w'
      write (67,*)  Nit,w

      do 100 i=0,N
           phi(i)=0.d0
           phinew(i)=0.d0
           x=dfloat(i)*h
           s(i)=12.d0*x*x
100   continue
      E=0.d0
      onemw=1.d0-w
      wo2=w/2.d0
      h2=h*h

      do 1000 it=1,Nit

      phinew(0)=onemw*phi(0)+wo2*(phi(1)+h2*s(0))
      do 200 i=1,N-1
           phinew(i)=onemw*phi(i)+wo2*((phi(i-1)+phi(i+1))
     1           +h2*s(i) )
200   continue
      phinew(N)=onemw*phi(N)+wo2*(phi(N-1)+h2*s(N))

      do 300 i=1,N-1
           phi(i)=phinew(i)
300   continue
      if (N.le.10) write (68,888) (phi(i),i=0,N)
888   format(11f7.4)

      E=0.d0
      do 400 i=1,N-1
           E=E+0.5d0*(phi(i+1)-phi(i))*(phi(i+1)-phi(i))
     1           -h2*s(i)*phi(i)
400   continue

      write (67,990) it,E/h
      write ( 6,990) it,E/h
990   format(i8,f12.8)

1000  continue

      write (67,*) ' '
      do 1100 i=0,N
           write (67,991) i,dfloat(i)*h,phi(i)
1100  continue
991   format(i8,2f12.8)
992   format(2f12.8)

      h=0.005
      do 1200 i=0,200
           x=dfloat(i)*h
           write (67,992) x,x-x*x*x*x
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1200      continue
          end
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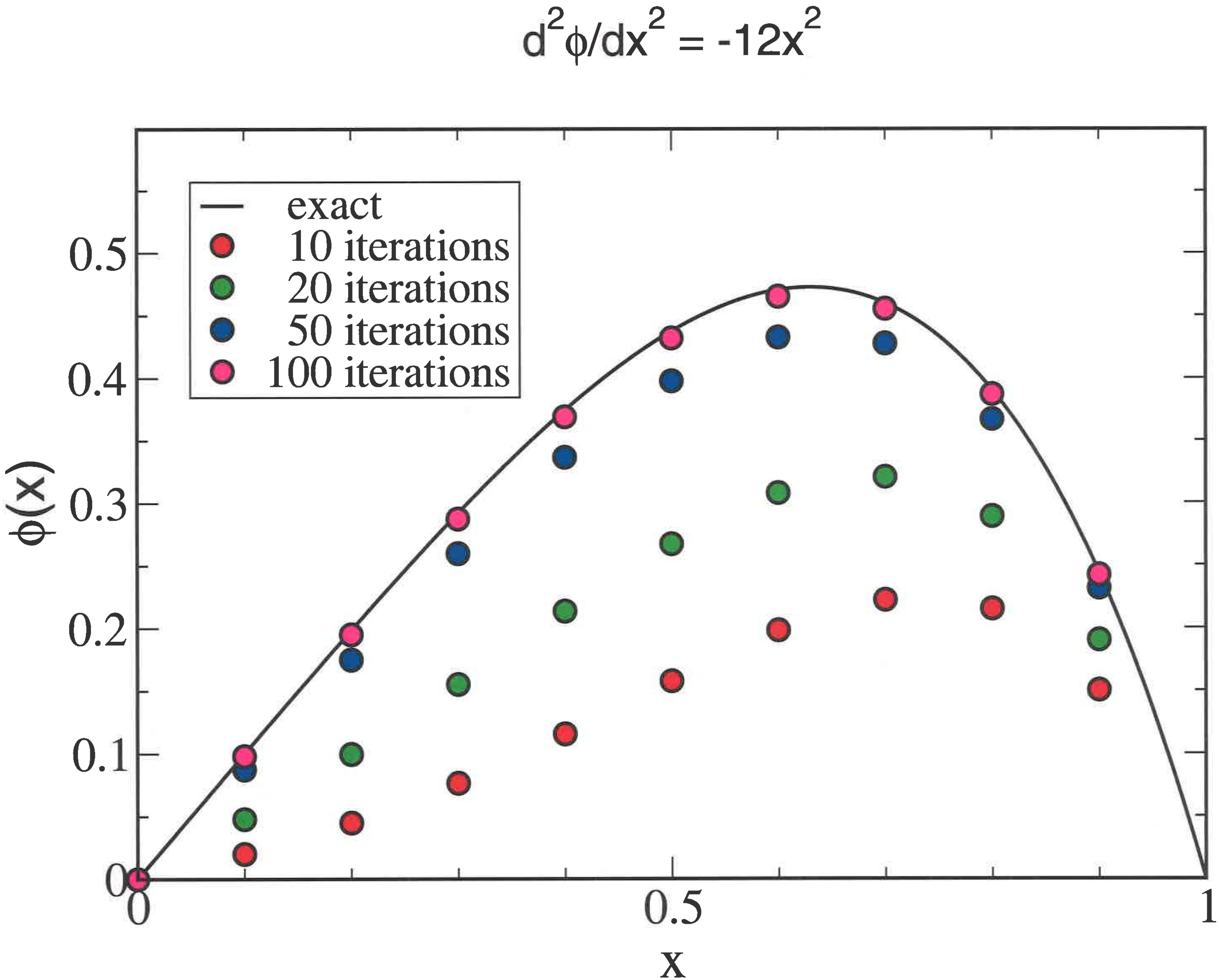

$d^2\phi/dx^2 = -12x^2$
exact
10 iterations
20 iterations
50 iterations
100 iterations
0.5
0.4
0.3
0.2
0.1
0
$\phi(x)$
0
0.5
1
x