PHYSICS 110A, WINTER 2017
ELECTRICITY AND MAGNETISM

Assignment Four, Due Saturday, February 10, 5:00 pm.

[1.] Find the potential at the point $(0, 0, z)$ due to three charges: $+q$ at $(0, a, 0)$, $+q$ at $(0, -a, 0)$, and $-2q$ at $(0, 0, -b)$. Using Coulomb's law, obtain the electric field $\mathbf{E}$. Can you get $\mathbf{E}$ from the potential? If yes, does it agree with Coulomb's law?

[2.] How much work does it take to assemble the charge distribution of problem [1.]?

[3.] Griffiths Problem 25, Chapter 2.

[4.] Griffiths Problem 27, Chapter 2.

[5.] Griffiths Problem 30, Chapter 2.

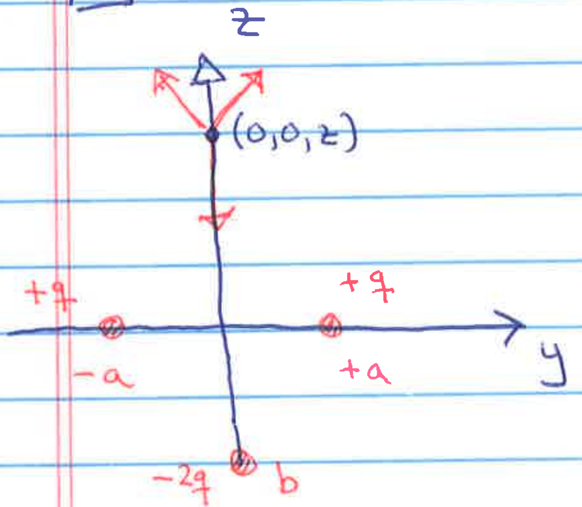
[6.] Griffiths Problem 34, Chapter 2.

[7.] Griffiths Problem 39, Chapter 2.

[8.] Griffiths Problem 43, Chapter 2.

Physics 110A
Assignment Four Solutions

$\boxed{1}$



$$V = \frac{2q}{\sqrt{z^2+a^2}} - \frac{2q}{b+z}$$

$$\vec{E} = -\vec{\nabla} V$$

no x, y dependence so $E_x = E_y = 0$ which is obviously true from vector sum of 3 $\vec{E}$ fields

$$\vec{E} = \hat{z}\left(-\frac{\partial}{\partial z}\right)\left(\frac{2q}{\sqrt{z^2+a^2}} - \frac{2q}{b+z}\right)$$

$$= 2q\,\hat{z}\left(\frac{z}{(z^2+a^2)^{3/2}} - \frac{1}{(b+z)^2}\right)$$

$\boxed{2}$ Work to assemble is $\sum_{i>j} q_i q_j / 4\pi\epsilon_0 r_{ij}$

so $$q^2/4\pi\epsilon_0(2a) - 2\left(\frac{q(2q)}{\sqrt{a^2+b^2}}\right)\frac{1}{4\pi\epsilon_0}$$

$$= q^2/4\pi\epsilon_0\left\{\frac{1}{2a} - \frac{4}{\sqrt{a^2+b^2}}\right\}$$

2.

$\boxed{3}$ Griffiths 2-25

(a) $V = \frac{1}{4\pi\epsilon_0} \, 2 \, q / \sqrt{z^2 + d^2/4}$

$$\vec{E} = -\vec{\nabla} V = -\hat{z} \, q/2\pi\epsilon_0 \, \partial/\partial z \, (z^2 + d^2/4)^{-1/2}$$

$$= qz / 2\pi\epsilon_0 (z^2 + d^2/4)^{3/2} \, \hat{z}$$

agrees with example #1

ie only along $\hat{z}$ axis

Important note: We are obtaining only $V(0,0,z)$.

We have not (because lack of symmetry makes it harder)

computed $V(x,y,z)$ at arbitrary points that are

"off center". Thus we really should be careful about

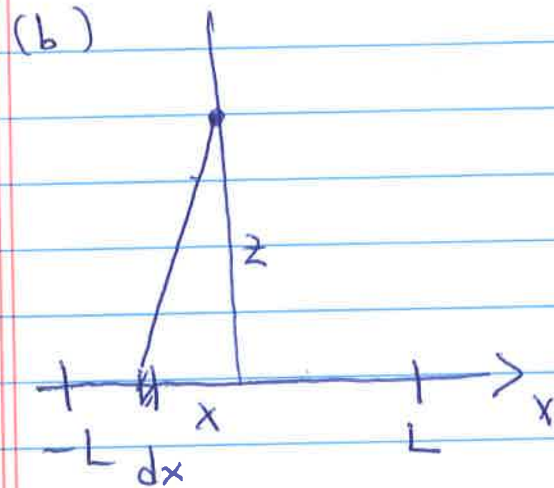
concluding $E_x = E_y = 0$. It is true in this case, but

false in the case where one $+q$ becomes $-q$

(this is answer to later part of problem)

3.

(b)



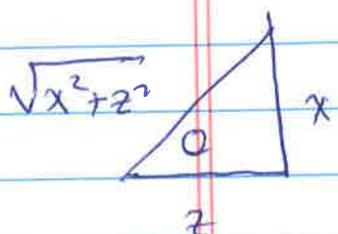
$$V = \frac{1}{4\pi\epsilon_0}\int_{-L}^{L} \frac{\lambda\, dx}{(x^2+z^2)^{1/2}}$$

$$x = z\tan\theta \qquad dx = z\sec^2\theta\, d\theta$$

$$= \frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{L} \frac{z\sec^2\theta\, d\theta}{z\sec\theta}$$

$$V = \frac{\lambda}{4\pi\epsilon_0}\int \sec\theta = \frac{\lambda}{4\pi\epsilon_0}\ln(\sec\theta + \tan\theta)$$

you should memorize this! often occurs! (and tricky)



$$V = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\sqrt{x^2+z^2}}{z} + \frac{x}{z}\right)\Bigg|_{-L}^{L}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{L+(L^2+z^2)^{1/2}}{-L+(L^2+z^2)^{1/2}}\right]$$

Check $\vec{E} = -\vec{\nabla}V = -\hat{z}\,\frac{\lambda}{4\pi\epsilon_0}\left\{\frac{1}{L+(L^2+z^2)^{1/2}}\,\frac{1}{2}(L^2+z^2)^{-1/2}\,2z - \frac{1}{-L+(L^2+z^2)^{1/2}}\,\frac{1}{2}(L^2+z^2)^{-1/2}\,2z\right\}$

$$= \frac{-\lambda z}{4\pi\epsilon_0}\,\hat{z}\,\frac{1}{\sqrt{L^2+z^2}}\left\{\frac{-L+\sqrt{L^2+z^2} - L - \sqrt{L^2+z^2}}{L^2+z^2-L^2}\right\}$$

$$= \frac{\lambda L\hat{z}}{2\pi\epsilon_0}\,\frac{1}{z\sqrt{L^2+z^2}}$$

agrees w/ Ex 2!

4.

3 cont'd

(c) $$V = \int_0^R \frac{1}{4\pi\epsilon_0} \frac{\overbrace{2\pi r\,dr\,\sigma}^{dq}}{(r^2+z^2)^{1/2}}$$

$$= \frac{\sigma}{2\epsilon_0} (r^2+z^2)^{1/2}\Big|_0^R$$

$$= \frac{\sigma}{2\epsilon_0}\left\{\sqrt{R^2+z^2} - z\right\}$$

$$\vec{E} = -\vec{\nabla} V = -\hat{z}\frac{\sigma}{2\epsilon_0}\left(\frac{z}{\sqrt{R^2+z^2}} - 1\right)$$

As $z \to 0$ $\quad \vec{E} = \sigma/2\epsilon_0\,\hat{z}$ $\quad$ (∞ plane result ✓✓)

As $z \to \infty$ $\quad \vec{E} = -\hat{z}\frac{\sigma}{2\epsilon_0}\left\{(1+R^2/z^2)^{-1/2} - 1\right\}$

$$\left\{1 - R^2/2z^2 - 1\right\}$$

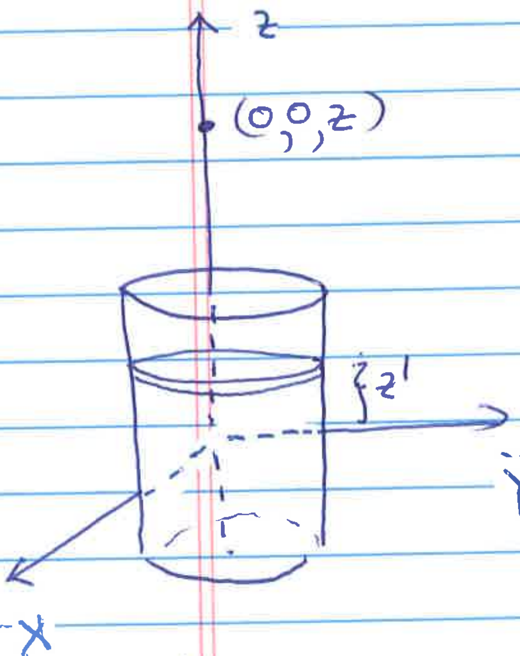
$$\vec{E} = \frac{\sigma R^2}{4\epsilon_0}\hat{z} = \frac{\pi R^2\sigma}{4\pi\epsilon_0 z^2}\hat{z}$$ ✓

point charge $q = \pi R^2\sigma$ result

If we change $+q \to -q$ in (a) we get $V=0$

But we know $\vec{E} \neq 0$ $\vec{E}$ is in $+\hat{x}$ direction. The point is that we have only computed V on the $\hat{z}$ axis $V(0,0,z)$. If we got $V(x\,y\,z)$ at all points in space we could correctly obtain $\vec{E} = -\vec{\nabla}V$.

$\boxed{4}$ Griffiths 2-27



length L
radius R
charge density ρ

Divide into disks $dq = \pi R^2 dz\, \rho$

Use result of problem #3c with $\sigma = \rho dz'$

$$V = \int_{-L/2}^{L/2} \frac{\rho dz'}{2\epsilon_0} \left\{ \left[R^2 + (z-z')^2\right]^{1/2} - (z-z') \right\}$$

To do first integral $z - z' = R\tan\theta$

$$dz' = -R\sec^2\theta\, d\theta$$

$$\frac{\rho}{2\epsilon_0} \int -R\sec^2\theta\, d\theta \; R\sec\theta$$

$$\frac{\rho R^2}{2\epsilon_0} \int \sec^3\theta\, d\theta$$

To do this set $u = \sec\theta \qquad dv = \sec^2\theta\, d\theta \qquad \begin{cases} du = \sec\theta\tan\theta\, d\theta \\ v = \tan\theta \end{cases}$

$$\int \sec^3\theta\, d\theta = \int u\, dv = uv - \int v\, du$$

$$= \sec\theta\tan\theta - \int \tan^2\theta \sec\theta\, d\theta$$

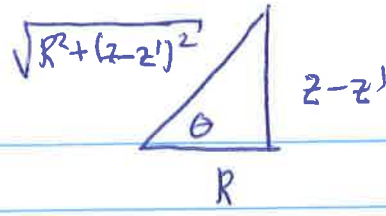
$$= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta\, d\theta$$

$$= \sec\theta\tan\theta + \ln(\sec\theta + \tan\theta) - \underbrace{\int \sec^3\theta\, d\theta}_{\text{starting pt!}}$$

$$\therefore \int \sec^3\theta\, d\theta = \frac{1}{2}\left[\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)\right]$$

6.

4 cont'd



We therefore get

$$V = \frac{\rho R^2}{4\epsilon_0}\left\{ \underbrace{\frac{\sqrt{R^2+(z-z')^2}}{R}}_{\sec\theta}\ \underbrace{\frac{z-z'}{R}}_{\tan\theta} + \ln\left(\frac{\sqrt{R^2+(z-z')^2}}{R} + \frac{z-z'}{R}\right)\right\}_{-L/2}^{L/2}$$

$$+\ \rho/2\epsilon_0 \left\{ -zz' + z'^2/2 \right\}_{-L/2}^{L/2} \qquad \leftarrow \quad \text{"easy"}\ -(z-z')\ \text{piece}$$

$$-\ \rho/2\epsilon_0\ Lz$$

Ugh.. a mess!

$$\vec{E} = -\vec{\nabla} V = -\hat{z}\frac{\partial}{\partial z} V$$

this is a tedious exercise in differentiation. Since differentiation involves no imagination, I just quote the result

$$\vec{E} = -\frac{\hat{z}\rho}{\epsilon_0}\left\{ \left[R^2+(z+L/2)^2\right]^{1/2} - \left[R^2+(z-L/2)^2\right]^{1/2} - L \right\}$$

Let's at least check $z \gg R, L$ limit.

$$\left[R^2+(z+L/2)^2\right]^{1/2} = \left[z^2+zL+L^2/4+R^2\right]^{1/2}$$

$$\approx z\left[1+\frac{L}{z}+\frac{L^2}{4z^2}+\frac{R^2}{z^2}\right]^{1/2}$$

7.

$$(1+x)^{1/2} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \dots$$

$$= z\left[1 + \frac{L}{2z} + \frac{L^2}{8z^2} + \frac{R^2}{2z^2} - \frac{1}{8}\left(\frac{L}{z} + \frac{L^2}{4z^2} + \frac{R^2}{z^2}\right)^2 + \dots\right]$$

$$\underbrace{-\frac{L^2}{8z^2} - \frac{L^3}{16z^3} - \frac{LR^2}{4z^3}}_{\text{to order } 1/z^3}$$

Changes sign in $(z - L/2)^2$ term

Putting together

$$\vec{E} = -\frac{\hat{z}\rho}{2\epsilon_0}\left\{ z + \frac{L}{2} + \frac{L^2}{8z} + \frac{R^2}{2z} - \frac{L^2}{8z} - \frac{L^3}{16z^2} - \frac{LR^2}{4z^2} - \left(z - \frac{L}{2} + \frac{L^2}{8z} + \frac{R^2}{2z} - \frac{L^2}{8z} - \frac{L^3}{16z^2} + \frac{LR^2}{4z^2}\right) - L \right\}$$

odd in L terms change sign

$$= \frac{\hat{z}\rho}{2\epsilon_0}\frac{LR^2}{2z^2} = \frac{1}{4\pi\epsilon_0}\frac{\pi R^2 L\rho}{z^2}$$

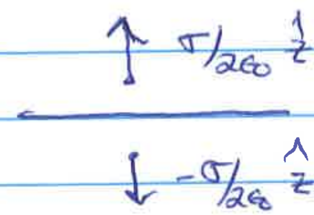
But $Q = \pi R^2 L\rho$ so this is

just point charge $Q/4\pi\epsilon_0 z^2$ result!

$\boxed{5}$ Griffiths 2-30

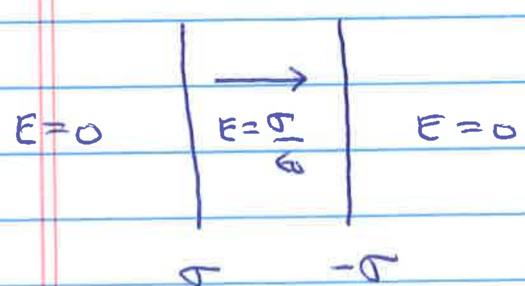
(a) Eqn 33 tells us $\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}$
about discontinuity in
$\vec{E}$ at a surface with
charge density σ

In Example 5, an infinite plane:



so get a difference σ/ϵ_0 ✓✓

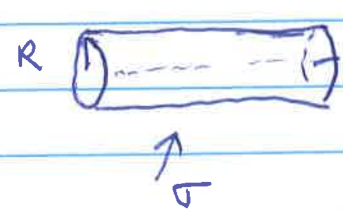
In Example 6



so differences are also σ/ϵ_0
at each plane

In problem 11 $\vec{E} = 0$ inside shell and $E = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^2\sigma}{R^2} = \frac{\sigma}{\epsilon_0}$
just outside shell (by Gauss' law) ✓✓

(b)



inside $E=0$
outside $2\pi r L E = 2\pi R L \sigma/\epsilon_0$
(distance r from axis) $E = \frac{R\sigma}{r\epsilon_0}$

for $r=R$ $E = \sigma/\epsilon_0$

$\boxed{6}$ Griffiths 2-34

a) Redo problem 21

outside $\quad r > R \qquad V(r) = \dfrac{Q}{4\pi\epsilon_0}\dfrac{1}{r}$

inside $\quad r < R \qquad$ Gauss' law gives

$$4\pi r^2 E = \tfrac{4}{3}\pi r^3 \rho / \epsilon_0 \qquad E = \rho r / 3\epsilon_0$$

So $\quad V(r) = \underbrace{\dfrac{Q}{4\pi\epsilon_0}\dfrac{1}{R}}_{\infty \text{ to } R} - \underbrace{\int_R^r \dfrac{\rho r'}{3\epsilon_0} dr'}_{R \text{ to } r}$

$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{R} - \frac{\rho r'^2}{6\epsilon_0}\Big|_R^r$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{R} - \underbrace{\frac{3Q}{4\pi R^3}}_{\rho}\frac{1}{6}\frac{1}{\epsilon_0}\left(r^2 - R^2\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{R} + \frac{Q}{8\pi\epsilon_0}\frac{1}{R} - \frac{Q}{8\pi\epsilon_0}\frac{r^2}{R^3}$$

$$= \frac{Q}{8\pi\epsilon_0 R}\left\{3 - r^2/R^2\right\}$$

$$= \frac{\rho R^2}{6\epsilon_0}\left\{3 - r^2/R^2\right\}$$

6 cont'd

Now do the Energy stored via Eqn 43

$$W = \frac{1}{2}\int \rho V d\tau = \frac{1}{2}\int_0^R 4\pi r^2 dr\, \rho\, \frac{\rho R^2}{6\epsilon_0}\left(3 - r^2/R^2\right)$$

nonzero only inside sphere

$$= \frac{\pi \rho^2 R^2}{3\epsilon_0}\int_0^R \left(3r^2 - r^4/R^2\right) dr$$

$$r^3 - r^5/5R^2 \Big|_0^R = \frac{4}{5}R^3$$

$$= \frac{4\pi\rho^2 R^5}{15\epsilon_0} = \frac{3}{20\pi\epsilon_0} Q^2/R$$

$$\rho = 3Q/4\pi R^3$$

b) Using Eq 45

$$W = \frac{1}{2}\epsilon_0\left[\int_0^R 4\pi r^2 dr\left(\frac{\rho r}{3\epsilon_0}\right)^2 + \int_R^\infty \left(\frac{R^3\rho}{3\epsilon_0}\frac{1}{r^2}\right)^2 4\pi r^2 dr\right]$$

(inside) (outside)

$$= \frac{\rho^2}{2\epsilon_0}\left[\frac{4\pi}{9}\frac{r^5}{5}\Big|_0^R + -\frac{4\pi R^6}{9}\frac{1}{r}\Big|_R^\infty\right]$$

$$\frac{4\pi}{9}\frac{R^5}{5} + \frac{4\pi}{9}R^5$$

$$= \frac{4\pi\rho^2 R^5}{15\epsilon_0}\ \checkmark \qquad \frac{4\pi}{9}R^5\frac{6}{5}$$

Interesting! ← 5 times as much outside!

12.

volume terms

c) Using Eq 44

assume $a > R$

$$W = \frac{1}{2}\epsilon_0 \left\{ \int_0^R 4\pi r^2 dr \left(\frac{\rho r}{3\epsilon_0}\right)^2 + \int_R^a 4\pi r^2 dr \left(\frac{R^3 \rho}{3\epsilon_0 r^2}\right)^2 \right.$$

$$\left. + \quad 4\pi a^2 \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\frac{Q}{4\pi\epsilon_0}\frac{1}{r^2} \right\}$$

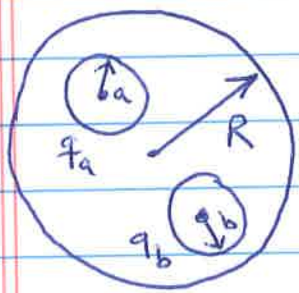
with $r = a$

surface term

pretty clearly the surface term just compensates for the upper limit being a rather than ∞ in the second volume term, so result is same as (b).

$\boxed{7}$ Griffiths 2-39

(a) σ_a must neutralize q_a so that $\vec{E} = 0$ inside the conductor



$$\sigma_a 4\pi a^2 = -q_a \qquad \sigma_a = -q_a / 4\pi a^2$$

similarly $\sigma_b = -q_b / 4\pi b^2$

The minus signs make sense: $+q_a, +q_b$ attract negative charge

The charge accumulated on small interior cavities must appear on the exterior surface

$$4\pi R^2 \sigma_R = (q_a + q_b) \qquad \sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

(b) Gauss' Law $(r > R)$ $\quad \vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q_a + q_b}{r^2} \hat{n}$

(c) Gauss' Law again cavity a $\quad \vec{E}_a = \frac{1}{4\pi\epsilon_0} \frac{q_a}{r^2} \hat{r}$

distance from cavity center

radial from cavity center

(d) zero

(e) σ_R changes (no longer uniform)
$\vec{E}_{outside}$ changes ; $\vec{E}_a, \vec{E}_b$ unchanged. $\vec{F}_{q_a} = \vec{F}_{q_b} = 0$

14.

$\boxed{8}$ Griffiths 2-43 Apply Gauss Law to cylindrical surface $a<r<b$

$$\underbrace{E\, 2\pi r L}_{\phi_E} = \lambda L / \epsilon_0 \qquad \lambda = \text{charge/length}$$

$$E = \lambda / 2\pi \epsilon_0 r$$

$$V_{ab} = \int_a^b \frac{\lambda}{2\pi\epsilon_0 r}\, dr = \frac{\lambda}{2\pi\epsilon_0} \ln b/a$$

$$C = Q/V = \lambda L / \lambda/2\pi\epsilon_0 \ln b/a$$

$$C = 2\pi\epsilon_0 L / \ln(b/a)$$