PHYSICS 110A, WINTER 2017
ELECTRICITY AND MAGNETISM

Assignment Three, Due Friday, February 2, 5:00 pm.

[1.] Consider a wire of length a and uniform linear charge density λ. Compute the electric field $\mathbf{E}$ along the axis of the wire. Consider both the case when you are evaluating $\mathbf{E}$ at points outside the wire, and for points inside the wire.

[2.] Compute the electric field at the center of a thin circular ring of radius R if the ring is divided into two semicircles (by a thin piece of insulator) and has charge per unit length $+\lambda_1$ on one semicircle and $-\lambda_2$ on the other semicircle.

[3.] Griffiths Problem 5, Chapter 2.

[4.] Griffiths Problem 7, Chapter 2.

[5.] Griffiths Problem 8, Chapter 2.

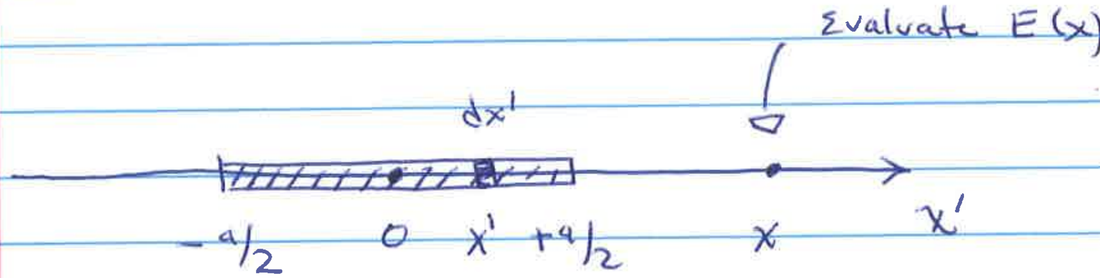
[6.] Griffiths Problem 9, Chapter 2.

[7.] Griffiths Problem 14, Chapter 2.

1.

Physics 110A
Assignment 3 Solutions

$\boxed{1}$



$$E_x = \int_{-a/2}^{a/2} \frac{1}{4\pi\epsilon_0} \frac{\lambda\, dx'}{(x-x')^2}$$

(Assume $x > a/2$)

Obviously $E_y = E_z$ along wire axis...

$$= \frac{\lambda}{4\pi\epsilon_0} \frac{1}{x-x'}\Bigg|_{-a/2}^{a/2}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \left\{ \frac{1}{x-a/2} - \frac{1}{x+a/2} \right\}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \frac{x+a/2 - x + a/2}{x^2 - a^2/4}$$

$$= \frac{\lambda a}{4\pi\epsilon_0} \frac{1}{x^2 - a^2/4}$$

In limit $x \gg a$ this goes to correct point charge result $E_x = \frac{Q}{4\pi\epsilon_0 x^2}$ since $Q = \lambda a$

1 cont'd On the other hand if $0 < x < a/2$

must take into account change in direction of E

$$E_x = \int_{-a/2}^{x} \frac{1}{4\pi\epsilon_0} \frac{\lambda dx'}{(x-x')^2} - \int_{x}^{a/2} \frac{1}{4\pi\epsilon_0} \frac{\lambda dx'}{(x-x')^2}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \left\{ \frac{1}{x-x'} \Big|_{-a/2}^{x} - \frac{1}{x-x'} \Big|_{x}^{a/2} \right\}$$

The apparent divergence at $x' = x$ cancels in the two expressions, leaving

$$= \frac{\lambda}{4\pi\epsilon_0} \left\{ -\frac{1}{x+a/2} - \frac{1}{x-a/2} \right\}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \left\{ \frac{-x+a/2 - x - a/2}{x^2 - a^2/4} \right\}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \frac{2x}{a^2/4 - x^2}$$

← write denominator this way since $a^2/4 - x^2 > 0$

Checks: dimensions okay ✓

$E_x = 0$ at $x = 0$ as expected ✓

Note E_x diverges as $x \to a/2$ (same as expression on page 1). Reasonable since you are approaching very close to charge

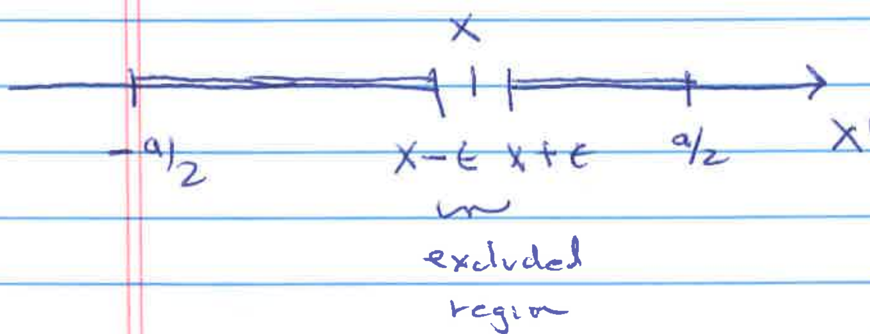
Note: Expressions for $x < 0$ are obvious by symmetry.

2A

Alternate ways of dealing with divergences in calculation of E in wire interior:

(1) Exclude a small region and take $\epsilon \to 0$

$$E_x = \int_{-a/2}^{x-\epsilon} \frac{1}{4\pi\epsilon_0}\frac{\lambda\, dx'}{(x-x')^2} - \int_{x+\epsilon}^{a/2} \frac{1}{4\pi\epsilon_0}\frac{\lambda\, dx'}{(x-x')^2}$$

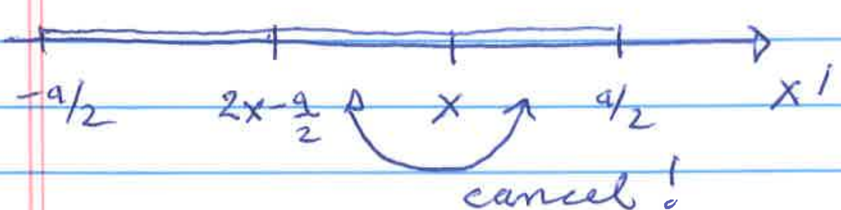


$$= \frac{\lambda}{4\pi\epsilon_0}\left\{ \frac{1}{x-x'}\Bigg|_{-a/2}^{x-\epsilon} - \frac{1}{x-x'}\Bigg|_{x+\epsilon}^{a/2} \right\}$$

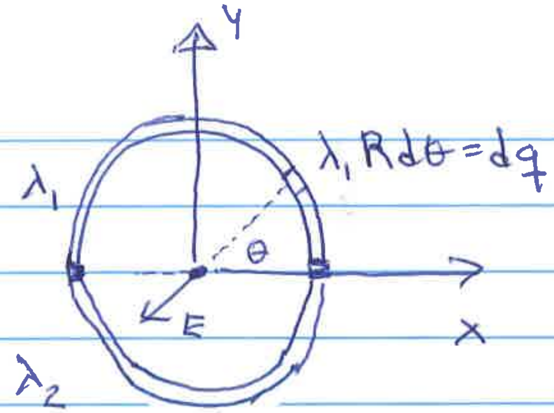
$$= \frac{\lambda}{4\pi\epsilon_0}\left\{ \left(\frac{1}{\epsilon} - \frac{1}{x+a/2}\right) - \left(\frac{1}{x-a/2} - \frac{1}{-\epsilon}\right) \right\}$$

cancellation is now clean!

(2) Argue that contributions from x to $a/2$ are cancelled by those from $2x - a/2$ to x (by symmetry) leaving only the contribution from $-a/2$ to $2x - a/2$



$\boxed{2}$



Clearly by symmetry $E_x = 0$ and we need to compute E_y. Do it for λ_1 and λ_2 and combine.

$$E_y^{(1)} = -\int_0^{\pi} \frac{\lambda_1 R\,d\theta}{4\pi\epsilon_0 R^2} \sin\theta$$

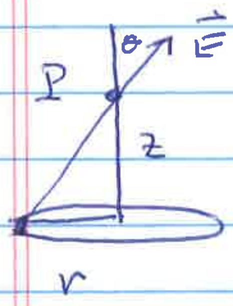
to get E_y

$$= \frac{-\lambda_1}{4\pi\epsilon_0} \frac{1}{R} (-\cos\theta)\Big|_0^{\pi} = \frac{-\lambda_1}{2\pi\epsilon_0 R}$$

Clearly the value for $E_y^{(2)} = +\lambda_2/2\pi\epsilon_0 R$

$$E_y = \frac{\lambda_2 - \lambda_1}{2\pi\epsilon_0 R}$$

$\boxed{3}$ Griffiths 2-5



All points on the loop are $(r^2+z^2)^{1/2}$ from P. Likewise all $\vec{E}$ have the same angle θ with respect to $\hat{z}$ axis.

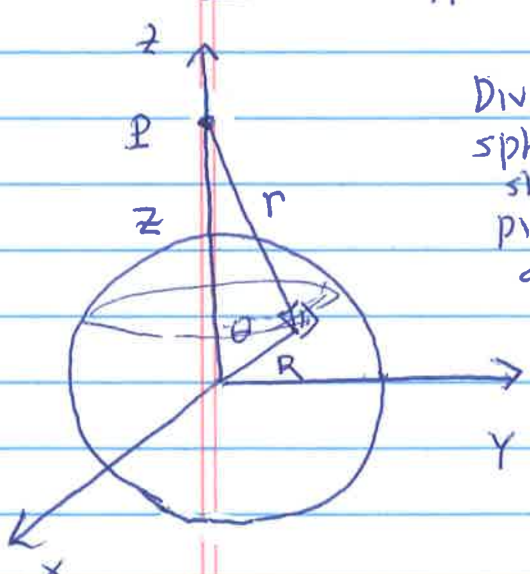
$\therefore \quad E_x = E_y = 0 \quad$ (symmetry)

$$E_z = \frac{(\lambda 2\pi r)}{4\pi\epsilon_0} \frac{1}{r^2+z^2} \frac{z}{(r^2+z^2)^{1/2}}$$

$$= \frac{\lambda r z}{2\epsilon_0} \frac{1}{(r^2+z^2)^{3/2}}$$

NB we used this in getting $\vec{E}$ due to a disk in class!

$\boxed{4}$ Griffiths 2-7 Clearly $E_x = E_y = 0$



Divide spherical shell into pieces at angle θ

$$r^2 = R^2 + z^2 - 2Rz\cos\theta$$

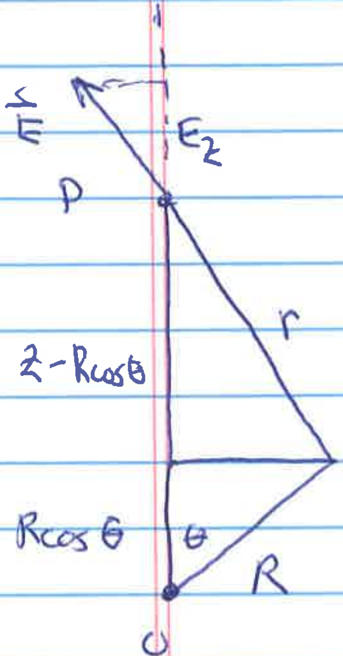
These are at distance r from P and carry a charge

$$\underbrace{2\pi R \sin\theta R d\theta}_{dA} \; \sigma$$

Check $\displaystyle\int_0^\pi dA = \int_0^\pi 2\pi R^2 \sin\theta \, d\theta$

$$= 2\pi R^2 (-\cos\theta)\Big|_0^\pi = 4\pi R^2 \checkmark$$

4 cont'd



$$E_z = \frac{1}{4\pi\epsilon_0}\int_0^{\pi} \frac{2\pi R^2 \sin\theta\, \sigma\, d\theta}{R^2+z^2-2Rz\cos\theta}\;\frac{z - R\cos\theta}{(R^2+z^2-2Rz\cos\theta)^{1/2}}$$

to get E_z

$$= \frac{R^2\sigma}{2\epsilon_0}\left\{\int_0^{\pi} \frac{z\sin\theta\, d\theta}{(R^2+z^2-2Rz\cos\theta)^{3/2}} - \int_0^{\pi} \frac{R\cos\theta\sin\theta\, d\theta}{[R^2+z^2-2Rz\cos\theta)^{3/2}}\right\}$$

Let $u = \cos\theta$ $\quad du = -\sin\theta\, d\theta$

$$E_z = \frac{R^2\sigma}{2\epsilon_0}\left\{\int_1^{-1} \frac{-z\,du}{(R^2+z^2-2Rzu)^{3/2}} - \int_1^{-1} \frac{-Ru\,du}{(R^2+z^2-2Rzu)^{3/2}}\right\}$$

$$\frac{(-2)z}{-2Rz}\left(R^2+z^2-2Rzu\right)^{-1/2}\Big|_{-1}^{1}$$

$$\frac{1}{R}\left\{\frac{1}{|R-z|} - \frac{1}{R+z}\right\}$$

take positive square root!

Define $a = R^2+z^2$

$b = 2Rz$

$$-\frac{bu}{2z}\,du \Big/ (a-bu)^{3/2}$$

$$\frac{1}{2z}\;\frac{-a+a-bu\quad du}{(a-bu)^{3/2}}$$

6.

First contribution $\frac{R^2\sigma}{2\epsilon_0}\frac{1}{R}\left\{\frac{1}{|R-z|}-\frac{1}{R+z}\right\}$

Second contribution (need to subtract from first → reverse limits and add)

$$\frac{1}{2z}\left\{-a(a-bu)^{-1/2}\left(\frac{-2}{-b}\right)+(a-bu)^{1/2}\left(\frac{2}{-b}\right)\right\}_{-1}^{+1}$$

$$=\frac{1}{2z}\left\{-\frac{2a}{b}\left(\frac{1}{\sqrt{a-b}}-\frac{1}{\sqrt{a+b}}\right)-\frac{2}{b}\left(\sqrt{a-b}-\sqrt{a+b}\right)\right\}$$

$$\sqrt{a-b}=R+z \qquad \sqrt{a-b}=|R-z|$$

$$=-\frac{1}{2Rz^2}\left\{(R^2+z^2)\left(\frac{1}{|R-z|}-\frac{1}{R+z}\right)+|R-z|-(R+z)\right\}$$

Put this all together for $z<R$ (interior)

$$\frac{R^2\sigma}{2\epsilon_0}\left\{\underbrace{\frac{1}{R}\frac{1}{R-z}-\frac{1}{R}\frac{1}{R+z}}_{\text{first}}-\frac{1}{2Rz^2}\left\{(R^2+z^2)\left(\frac{1}{R-z}-\frac{1}{R+z}\right)+R-z-R-z\right\}\right\}$$

$$R^2+z^2\ \frac{R+z-R+z}{R^2-z^2}-2z$$

$$\frac{1}{R}\ \frac{R+z-R+z}{R^2-z^2} \qquad -\frac{1}{Rz}\left\{\frac{R^2+z^2}{R^2-z^2}-1\right\}$$

$$-\frac{1}{Rz}\left\{\frac{R^2+z^2-R^2+z^2}{R^2-z^2}\right\}$$

$$+\frac{1}{R}\ \frac{2z}{R^2-z^2}$$

$$-\frac{1}{R}\left\{\frac{2z}{R^2-z^2}\right\}$$

Cancels to zero!!

$4\pi z^2 E_z = 4\pi R^2\sigma/\epsilon_0$

Similarly for $z>R$ get "obvious" Gauss law result $E_z=\frac{R^2\sigma}{\epsilon_0 z^2}$

(see page 7)

If $z > R$ (Exterior)

$$\frac{R^2\sigma}{2\epsilon_0}\left\{\underbrace{\frac{1}{R}\frac{1}{z-R} - \frac{1}{R}\frac{1}{R+z}}_{\text{first}} - \frac{1}{2Rz^2}\left\{(R^2+z^2)\left(\frac{1}{z-R} - \frac{1}{R+z}\right) + z-R-R-z\right\}\right.$$

$$\frac{1}{R}\frac{R+z-z+R}{z^2-R^2} \qquad - \frac{1}{2Rz^2}\left\{(R^2+z^2)\frac{R+z-z+R}{z^2-R^2} - 2R\right\}$$

$$+\frac{2}{z^2-R^2} \qquad - \frac{1}{z^2}\left\{\frac{R^2+z^2-z^2+R^2}{z^2-R^2}\right\}$$

$$- \frac{1}{z^2}\left\{\frac{2R^2}{z^2-R^2}\right\}$$

Put together

$$\frac{R^2\sigma}{2\epsilon_0}\left\{\frac{2}{z^2-R^2}\right\}\left\{1 - \frac{R^2}{z^2}\right\}$$

$$\frac{z^2-R^2}{z^2}$$

$$\frac{R^2\sigma}{\epsilon_0 z^2} \qquad !!$$

$\boxed{5}$ Griffiths 2-8

Sphere of uniform volume charge density ρ can be thought of as a collection of shells. The surface charge density σ is related to volume charge density via $\sigma = \rho\, dr$

Exterior $\quad E_z = \dfrac{r^2 \sigma}{\epsilon_0 z^2}$ for each shell of radius r

For sphere, then, $\quad E_z = \displaystyle\int_0^R \frac{r^2 \rho}{\epsilon_0 z^2}\, dr$

$$= \frac{R^3 \rho}{3\epsilon_0 z^2}$$

Check: Since $\quad Q = \frac{4}{3}\pi R^3 \rho \qquad E_z = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{z^2}$ ✓✓

agrees w/ Gauss

Interior $\quad E_z = 0$ inside shell, so only shells with $r < z$ contribute $\quad E_z = r^2 \sigma / \epsilon_0 z^2$

$$E_z = \int_0^z \frac{r^2 \rho\, dr}{\epsilon_0 z^2} = \frac{\rho}{3\epsilon_0} z$$

Gauss gives $\quad 4\pi z^2 E_z = \frac{4}{3}\pi z^3 \rho$ ✓✓

9.

$\boxed{6}$ Griffiths 2-9

$\vec{E} = kr^3\hat{r}$ ← E has units $\frac{1}{[\epsilon_0]}\frac{Q}{L^2} = [k]L^3$

(a) $\rho = \epsilon_0 \bar{\nabla}\cdot\bar{E} = \epsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 kr^3\right) = \epsilon_0 \frac{1}{r^2} 5kr^4$

$\rho = 5\epsilon_0 k r^2$

↑ units are $[\rho] = [\epsilon_0][k]L^2$

$= [\epsilon_0]\frac{Q}{[\epsilon_0]}\frac{1}{L^5}L^2 = \frac{Q}{L^3}$ ✓

(b) get Q by integrating ρ

$Q = \int_0^R \underbrace{4\pi r^2 dr}_{\text{shells of radius } r \text{ have this volume}} \rho = 4\pi\, 5\epsilon_0 k \int_0^R r^4 dr$

$= 4\pi\epsilon_0 k R^5$

—or— Gauss' law $\quad \Phi_E = 4\pi R^2 \underbrace{|\vec{E}|}_{kR^3} = 4\pi k R^5 = \frac{Q}{\epsilon_0}$

$\boxed{7}$ Griffiths 2-14

$$\phi_E = 4\pi r^2 |\vec{E}|$$

$$Q_{enclosed} = \int_0^r \underbrace{4\pi r'^2 dr'}_{\text{Volume of shell of radius } r'} \overset{kr'}{\rho} = 4\pi \cdot k \left.\frac{r'^4}{4}\right|_0^r$$

$$= \pi k r^4$$

$$\phi_E = Q_{enclosed}/\epsilon_0$$

$$4\pi r^2 |\vec{E}| = \frac{1}{\epsilon_0} \pi k r^4$$

$$|\vec{E}| = \frac{k}{4\epsilon_0} r^2 \qquad \vec{E} = \frac{k}{4\epsilon_0} r^2 \hat{r}$$

✓✓

Units of k are $\frac{Q}{L^4}$ $\Rightarrow$ Units of $|\vec{E}| = \frac{Q}{[\epsilon_0] L^4} L^2 = \frac{Q}{[\epsilon_0] L^2}$

Can check by divergence theorem?

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 = kr/\epsilon_0$$

$$1/r^2 \frac{\partial}{\partial r} r^2 E_r = kr/\epsilon_0$$

$$\partial/\partial r \; r^2 E_r = kr^3/\epsilon_0$$

$$r^2 E_r = \frac{kr^4}{4\epsilon_0} \qquad E_r = \frac{kr^2}{4\epsilon_0}$$

✓✓