# PHYSICS 110A, WINTER 2017
# ELECTRICITY AND MAGNETISM

**Assignment Two, Due Friday, January 19, 5:00 pm.**

**[1.]** Compute the divergence of the vector field

$$\mathbf{v} = C\,\mathbf{r}$$

In a week or so we will prove this is the electric field inside a sphere with uniform charge density $\rho$, if we define $C = 4\pi k\rho/3$. (You probably saw this problem in lower division E&M.) Using that statement, discuss the consistency of your result with the Maxwell Equation $\nabla \cdot \mathbf{E} = 4\pi k\rho$ which tells us the divergence of $\mathbf{E}$ measures the charge density.

**[2.]** Compute the curl of the vector field

$$\mathbf{v} = C\,(x\,\hat{\mathbf{y}} - y\,\hat{\mathbf{x}})$$

In Physics 110B you will prove this is the magnetic field inside a long straight wire with uniform current density $J$, if we define $C = \mu_0 J/2$. (You probably saw this problem in lower division E&M.) Using that statement, discuss the consistency of your result with the Maxwell Equation $\nabla \times \mathbf{B} = \mu_0 J$ which tells us the curl of $\mathbf{B}$ measures the current density.

**[3.]** Griffiths Problem 54, Chapter 1.

**[4.]** Griffiths Problem 55, Chapter 1.

**[5.]** Griffiths Problem 56, Chapter 1.

**[6.]** Griffiths Problem 57, Chapter 1.

**[7.]** Griffiths Problem 61, Chapter 1.

**[8.]** It is straightforward to compute the gradient of a given scalar function. In this problem we attempt the reverse process. Verify the force

$$\mathbf{F} = (3x^2yz - 3y)\hat{\mathbf{x}} + (x^3z - 3x)\hat{\mathbf{y}} + (x^3y + 2z)\hat{\mathbf{z}}$$

is conservative, and find a potential $\phi$ such that $\mathbf{F} = -\nabla\phi$.

**[9.]** It is straightforward to compute the curl of a given vector function. In this problem we attempt the reverse process. Find the vector field $\mathbf{A}$ such that $\mathbf{v} = \nabla \times \mathbf{A}$ for

$$\mathbf{v} = (y + z)\hat{\mathbf{x}} + (x - z)\hat{\mathbf{y}} + (x^2 + y^2)\hat{\mathbf{z}}$$

**[10.]** Compute the integral of $\nabla \cdot \mathbf{v}$ over the region $x^2 + y^2 + z^2 \leq 25$ for the vector field

$$\mathbf{v} = (x^2 + y^2 + z^2)(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}})$$

1.

Physics 110A Assignment 2

Winter 2018

$\boxed{1}$ $\vec{V} = C\vec{r} = C(x\hat{x} + y\hat{y} + z\hat{z})$

$$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = C(1+1+1) = 3C$$

If $C = 4\pi k\rho/3$ $\quad \vec{\nabla}\cdot\vec{V} = 3\left(\frac{4\pi k\rho}{3}\right) = 4\pi k\rho$

which is consistent with Maxwell.

$\boxed{2}$ $\vec{V} = C(x\hat{y} - y\hat{x})$

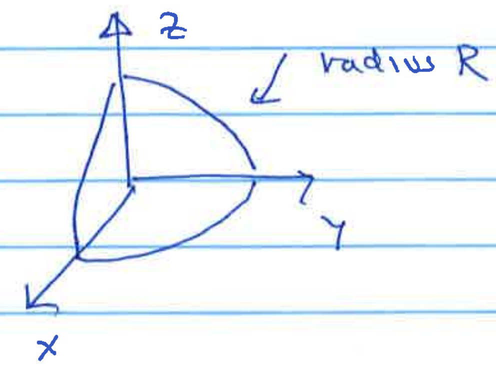
$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -yC & xC & 0 \end{vmatrix} = 2\hat{z}C$$

$|\vec{\nabla}\times\vec{V}| = 2C = 2(\mu_0 J/2) = \mu_0 J$ if $C = \frac{\mu_0 J}{2}$

This is consistent with $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$.

2.

$\boxed{3.}$ (Griffiths 54)




$$\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{v}\, d\tau = \int_{S} \vec{v}\cdot d\vec{a}$$

$$\vec{v} = (r^2\cos\theta)\hat{r} + (r^2\cos\phi)\hat{\theta} - r^2\cos\theta\sin\phi\ \hat{\phi}$$

Spherical coordinates:

$$\vec{\nabla}\cdot\vec{v} = 1/r^2\, \partial/\partial r\,(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

First do left hand side

$$\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{v}\, d\tau$$

$$= \int_{\mathcal{V}} 1/r^2 \frac{\partial}{\partial r}(r^4\cos\theta) + \underbrace{1/r\sin\theta\ r^2\cos\theta\cos\phi + 1/r\sin\theta\,(-r^2\cos\theta\cos\phi)}_{\text{cancel...}}$$

$$= \int_{\mathcal{V}} \frac{1}{r^2} 4r^3\cos\theta\, d\tau = \underbrace{\int_0^R r^2 dr \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\, d\theta}_{\int d\tau}\ \underbrace{\cos\theta\, 4r}_{\vec{\nabla}\cdot\vec{v}}$$

$$= \frac{\pi}{2}\cdot\frac{1}{2}\cdot\int_0^R 4r^3\, dr = \frac{\pi}{4}\, r^4\Big|_0^R = \frac{\pi}{4}R^4$$

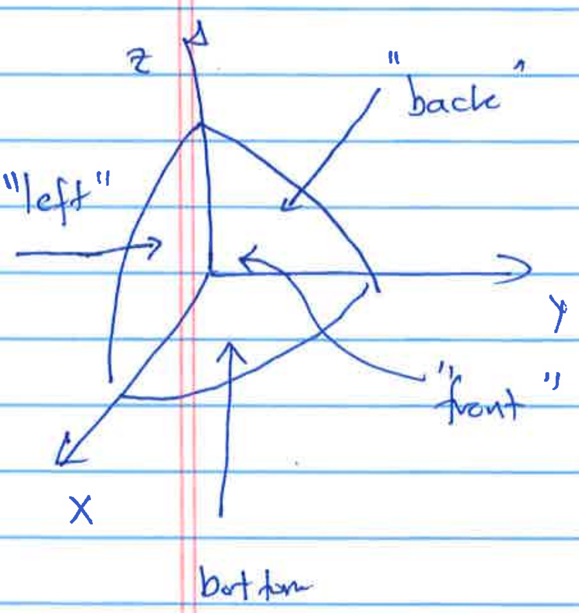
$\phi$ integral    $\theta$ integral

$$\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta = \frac{\sin^2\theta}{2}\Big|_0^{\pi/2} = 1/2$$

3.

(3 cont'd) Now tackle right hand side.

There are four surfaces




Along the "left" surface we have a normal equal to $-\hat{y}$.

We also have $\phi = 0$. Looking at Eq 64 for $\phi = 0$ we see

$$\hat{r} = \sin\theta\,\hat{x} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\theta\,\hat{x} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = \hat{y}$$

see "aside" for general principle

So the normal to the left surface is $-\hat{y} = -\hat{\phi}$.

(This can also be seen geometrically). Then, along "left"

$$\int_{S_{left}} \vec{v}\cdot\vec{da} = \int_{S_{left}} \underbrace{(-r^2\cos\theta\sin\phi)}_{v_\phi}\,da = \phi$$

but $\phi = 0$ so $\sin\phi = 0$.

(3 cont'd) Along the "back" the normal is $-\hat{x}$ and $\phi = \pi/2$

again using (64) we see
$$\hat{r} = \sin\theta\,\hat{y} + \cos\theta\,\hat{z}$$
$$\hat{\theta} = \cos\theta\,\hat{y} - \sin\theta\,\hat{z}$$
$$\hat{\phi} = -\hat{x}$$

once again normal is $\hat{\phi}$. (This can also be seen geometrically...)

$$\int_{S_{back}} \vec{v}\cdot\vec{da} = \int_{S_{back}} (\underbrace{-r^2\cos\theta}_{v_\phi}\,\underbrace{\sin\phi}_{1 \text{ since } \phi=\pi/2})|\vec{da}|$$

$S_{back}$ has $0<\theta<\pi/2$, $\phi = \pi/2$, $0<r<R$

$$\vec{da} = \underbrace{r\,r\,d\theta}_{|\vec{da}|}\,\hat{\phi}$$

$$\int_{S_{back}} \vec{v}\cdot\vec{da} = \int_0^R dr \int_0^{\pi/2} r\,d\theta\,(-r^2\cos\theta)$$

$$= -\underbrace{\int_0^R r^3\,dr}_{R^4/4}\,\underbrace{\int_0^{\pi/2}\cos\theta\,d\theta}_{1}$$

$$= -R^4/4$$

(3 cont'd) "bottom" surface is $\theta = \pi/2$

the direction of normal is $-\hat{z}$

$\hat{r} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$\hat{\theta} = -\hat{z}$ ← So normal is $\hat{\theta}$

$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

See "aside" for general principle

$$\int_{S_{bottom}} \vec{v}\cdot d\vec{a} = \int \underbrace{r^2\cos\phi}_{v_\theta}\, \underbrace{|d\vec{a}|}_{r\sin\theta\, d\phi\, dr}$$

$\sin\theta = 1$

$$= \int_0^R r\,dr \int_0^{\pi/2} d\phi \;\; r^2\cos\phi$$

$$= \underbrace{\int_0^R r^3\,dr}_{R^4/4} \; \underbrace{\int_0^{\pi/2} \cos\phi\, d\phi}_{1} \qquad = +R^4/4$$

"bottom" cancels "back"! "left" vanishes. So it all comes down to "front"

(3 cont'd) "front" has $r = R$ $\qquad \vec{da}' = \underbrace{r\sin\theta\, d\phi\, r\, d\theta}_{\text{see "Aside"}}\,\hat{r}$

$$\int_{S_{front}} \vec{V} \cdot \vec{da} = \int_{S_{front}} \underbrace{r^2 \cos\theta}_{V_r} \, |\vec{da}|$$

$$= \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \, r^2 \sin\theta \; r^2 \cos\theta \qquad r = R$$

$$= R^4 \underbrace{\int_0^{\pi/2} d\phi}_{\pi/2} \; \underbrace{\int_0^{\pi/2} \sin\theta \cos\theta \, d\theta}_{\frac{\sin^2\theta}{2}\Big|_0^{\pi/2} = 1/2}$$

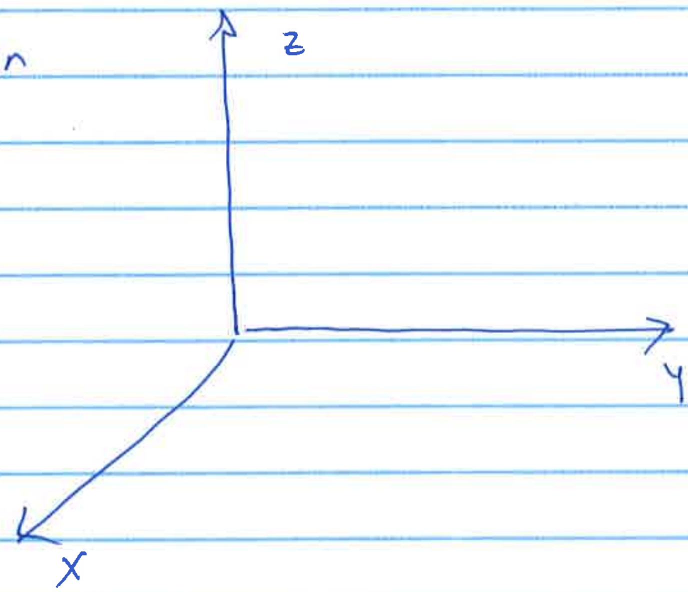
$$= \pi/4 \; R^4$$

At last!

ASIDE -1
7.

## Generally Useful Rule:

Cartesian




$x = \text{constant}$ : plane parallel to yz plane

normal is $\hat{x}$ $|\vec{da}| = dy\,dz$ $\vec{da} = \hat{x}\,dy\,dz$

Similarly:

$y = \text{constant}$ : $\vec{da} = \hat{y}\,dx\,dz$

$z = \text{constant}$ $\vec{da} = \hat{z}\,dx\,dy$

Notice unit vector giving direction of $\vec{da}$ in each case is same variable as that being held constant! The magnitude $|\vec{da}|$ is comprised of the "d" factors of the two variables which are not constant

ASIDE-2

8.

The same rule works in spherical coordinates ie

If $\quad \theta = \text{constant} \qquad \vec{da} = dr\, r\sin\theta\, d\phi \; \hat{\theta}$

$\phi = \text{constant} \qquad \vec{da} = dr\; r\,d\theta \; \hat{\phi}$

$r = \text{constant} \qquad \vec{da} = \underbrace{r\,d\theta\; r\sin\theta\, d\phi}_{\text{"d" factors from non constant variables}} \; \hat{r}$

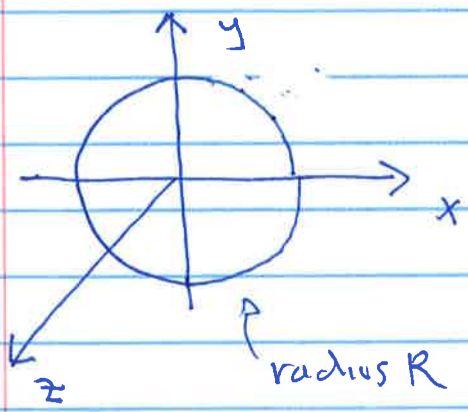
direction matches variable which is constant.

See if you can prove this prescription always works!

9.

$\boxed{4}$

(Griffiths 55) $\vec{v} = ay\,\hat{x} + bx\,\hat{y}$




$$\int_S \vec{\nabla}\times\vec{v}\cdot d\vec{a} = \int_P \vec{v}\cdot d\vec{l}$$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ay & bx & 0 \end{vmatrix} = (b-a)\,\hat{z}$$

$$d\vec{a} = \hat{z}\,dx\,dy$$ (assuming we go around P counterclockwise)

Since $\vec{\nabla}\times\vec{v}$ = constant $= b-a$

$$\int_S \vec{\nabla}\times\vec{v}\cdot d\vec{a} = (b-a)\underbrace{\pi R^2}_{\text{area of } S}$$

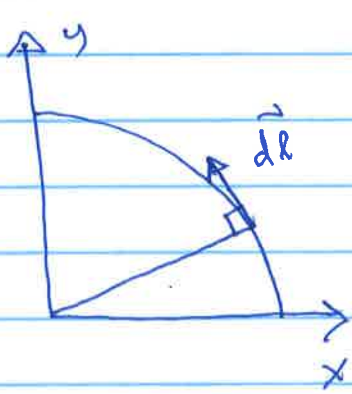
$$\int_P \vec{v}\cdot \underbrace{d\vec{l}}_{dx\,\hat{x}+dy\,\hat{y}} = \int_{-R}^{R} dx \quad \overset{v_x\,dx}{ay\,dx} + \underset{v_y}{bx}\underbrace{\left(\frac{-x}{\sqrt{R^2-x^2}}\right)dx}_{dy}$$

along P $\quad x^2+y^2=R^2$

so that $\quad y=(R^2-x^2)^{1/2}$

$$dy = \tfrac{1}{2}(R^2-x^2)^{-1/2}(-2x)\,dx$$

$$= -x/\sqrt{R^2-x^2}\;dx$$



10.

(4 cont'd) $\int_P \vec{v}\cdot d\vec{l} = \int_P a(R^2-x^2)^{1/2}\, dx - \dfrac{bx^2}{(R^2-x^2)^{1/2}}\, dx$

y

$x = R\sin\theta$

$(R^2-x^2)^{1/2} = R\cos\theta$

$dx = R\cos\theta\, d\theta$

$aR^2 \int_P \cos^2\theta\, d\theta \qquad - bR^2 \int_P \sin^2\theta\, d\theta$

to traverse P counterclockwise

$\theta$ goes from $2\pi$ to $\phi$

and $\cos^2\theta$ has average value $1/2$

same for $\langle \sin^2\theta \rangle = 1/2$

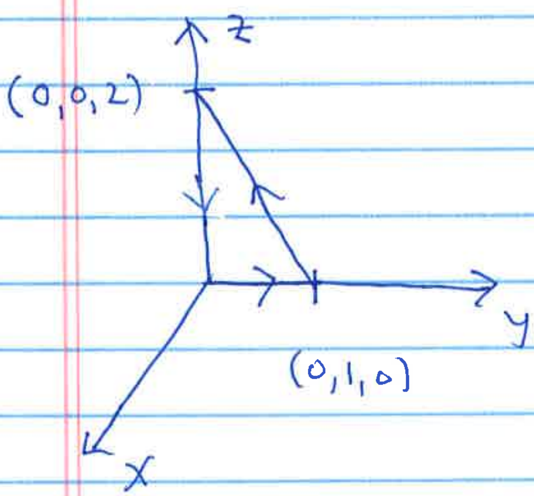
$\Rightarrow \quad aR^2(-2\pi)\tfrac{1}{2} \qquad -bR^2(-2\pi)\tfrac{1}{2}$

$\Rightarrow \quad -\pi a R^2 \qquad + \pi b R^2$

✓✓

11.

$\boxed{5}$ Griffiths 56 $\quad \vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$




$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6 & yz^2 & 3y+z \end{vmatrix}$$

$$= (3-2yz)\hat{x} + 0\hat{y} + 0\hat{z}$$

$$\vec{da} = dy\,dz\,\hat{x} \quad \text{(see "aside")}$$

$$\int_S \vec{\nabla}\times\vec{v}\cdot\vec{da} = \int_0^1 dy \int_0^{2-2y} dz\,(3-2yz)$$

$$= \int_0^1 dy \left\{ \underbrace{3(2-2y)}_{\int dz = z} - \underbrace{y(2-2y)^2}_{\int 2z = z^2} \right\}$$

$$= 6y - 3y^2 - 4\left(\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right)\Big|_0^1$$

$$= 6 - 3 - 4\underbrace{\left(1/2 - 2/3 + 1/4\right)}_{1/12(6-8+3)} = 6 - 3 - 1/3 = 8/3$$

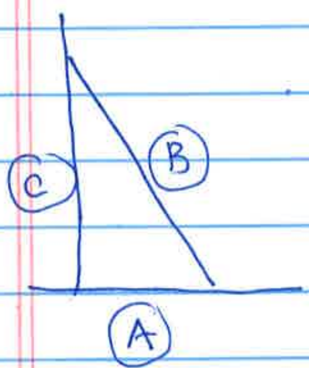
12.

$\boxed{5 \text{ cont'd}}$

A $\quad$ B $\quad z = 2-2y$, $\; dz = -2dy$

$$\int_P \vec{v}\cdot d\vec{l} = \int_0^1 y z^2\, dy + \int_1^0 yz^2\, dy + (3y+z)(-2dy)$$

~~$z = \phi$ along A~~




$$+ \int_2^0 (3y+z)\, dz$$

C $\quad y = 0$ along C

$$\int_1^0 y(2-2y)^2\, dy + \int_1^0 (3y+2-2y)(-2dy) + \int_2^0 z\, dz$$

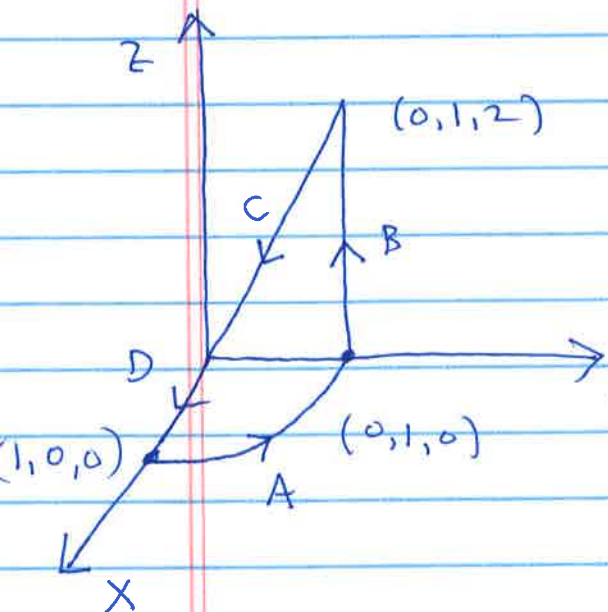
| | | |
|---|---|---|
| $= 4\left(\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right)\Big\vert_1^0$ | $-2 \quad 2y + y^2/2 \Big\vert_1^0$ | $z^2/2 \Big\vert_2^0$ |
| $= -4\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$ | $+2\left(2 + 1/2\right)$ | $-2$ |
| $= -4\frac{1}{12}(6-8+3)$ | | |
| $= -1/3$ | $+5$ | |

combines to $8/3$ again! ☒

13.

$\boxed{6}$ Griffiths 57

$$\vec{V} = r\cos^2\theta\,\hat{r} - r\cos\theta\sin\theta\,\hat{\theta} + 3r\,\hat{\phi}$$




PATH (A) $\theta = \pi/2$

$r = 1$

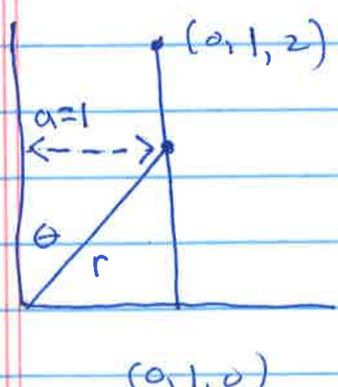
$0 < \phi < \pi/2$

$\vec{dl} = \hat{\phi}\,\underbrace{r\sin\theta}_{1 \text{ since } \theta=\pi/2}\,d\phi$

$$\int_{P_A} \vec{V}\cdot\vec{dl} = \int_0^{\pi/2} \underbrace{r\,d\phi\;3r}_{r=1 \text{ along } A} = 3\pi/2$$

convenient to keep track of units "length" a=1 units

PATH (B) $\phi = \pi/2$




$\sin\theta = a/r$

$\cos\theta\,d\theta = \frac{-a}{r^2}\,dr$

$1 < r < \sqrt{5}$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$= \sqrt{1-a^2/r^2}$

$\vec{dl} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta$

$= \hat{r}\,dr + \hat{\theta}\,r\left(\frac{-a\,dr}{r^2\cos\theta}\right)$

$\vec{dl}\cdot\vec{V}$

$= r\cos^2\theta\,dr$

$+ r\cos\theta\sin\theta\,\frac{dr}{r\cos\theta}$

(6 cont'd)

$$\int_{P_B} \vec{v}\cdot\vec{dl} = \int_1^{\sqrt{5}} v_r\,dr + r v_\theta\,d\theta \qquad -\frac{a\,dr}{r^2\cos\theta}$$

$$= \int_1^{\sqrt{5}} r\cos^2\theta\,dr - r\,r\cos\theta\sin\theta\,d\theta$$

$$= \int_1^{\sqrt{5}} r\left(1-\frac{a^2}{r^2}\right)dr + r^2\,\frac{a}{r}\,\frac{a\,dr}{r^2}$$

cancel

$$= \frac{r^2}{2}\Big|_1^{\sqrt{5}} = \frac{5-1}{2} = 2.$$

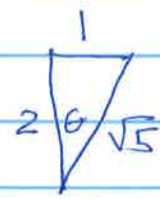
Path C

$\phi = \pi/2$

$\theta = \tan^{-1}\frac{1}{2}$

$0 < r < \sqrt{5}$

$\vec{dl} = \hat{r}\,dr$




$\cos\theta = 2/\sqrt{5}$

$\cos^2\theta = 4/5$

$$\int_{P_c} \vec{v}\cdot\vec{dl} = \int_{\sqrt{5}}^{0} r\cos^2\theta\,dr = \cos^2\theta\,\frac{r^2}{2}\Big|_{\sqrt{5}}^{0}$$

$$= -\frac{4}{5}\,\frac{5}{2} = -2$$

Path (D) $\theta = \pi/2$

$\phi = 0$

$0 < r < 1$

$\vec{d\ell} = \hat{r}\, dr$

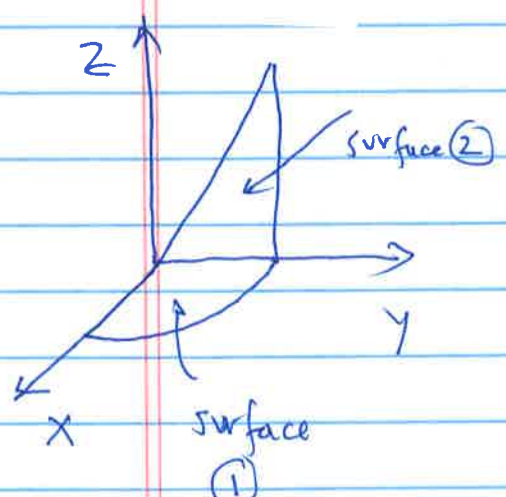
$$\int_{P_D} \vec{v}\cdot\vec{d\ell} = \int_0^1 r\cos^2\theta\, dr$$

$\phi$ since $\theta = \pi/2$

Adding $\frac{3\pi}{2} + 2 - 2 + \phi = \frac{3\pi}{2}$.

Now we need to do with $\vec{\nabla}\times\vec{v}$!

Two surfaces!




surface (1) has $\theta = \pi/2$

so (see "aside") the normal $\vec{da}$ is in $\hat{\theta}$ direction ← Actually $-\hat{\theta}$ by right hand rule

we need $(\vec{\nabla}\times\vec{v})_\theta$

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r} r v_\phi\right]$$

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} r\cos^2\theta - \frac{\partial}{\partial r} 3r^2\right]$$

$$= \frac{1}{r}(-6r) = -6$$

$(\vec{\nabla}\times\vec{v})_\theta$ is constant on surface (1) so

$$\int \vec{\nabla}\times\vec{v}\cdot d\vec{a} = +6\,(\text{Area}) = +6\,\frac{1}{4}\pi(1)^2 = \frac{3\pi}{2}$$

Consider, finally, surface (2). It has $\phi = \pi/2$

So, from "aside" normal $d\vec{a}$ is in $\hat{\phi}$ direction.

we need $(\vec{\nabla}\times\vec{v})_\phi$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r} r v_\theta - \frac{\partial v_r}{\partial\theta}\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(-r^2\cos\theta\sin\theta\right) - \frac{\partial}{\partial\theta} r\cos^2\theta\right]$$

$$= \frac{1}{r}\left[-2r\cos\theta\sin\theta + 2r\cos\theta\sin\theta\right] = 0$$

So we get $3\pi/2$ which checks!

$\boxed{7}$ (Griffiths 61)

a) If $\vec{v}(\vec{r}) = \vec{c}\, T(r)$ ← scalar function

$\vec{c}$: constant vector, indep of $\vec{r}$

$$\vec{\nabla}\cdot\vec{v} = T(r)\underbrace{\vec{\nabla}\cdot\vec{c}}_{\phi} + \vec{c}\cdot\vec{\nabla}T$$

$$\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{v}\, d\tau = \int \vec{c}\cdot\vec{\nabla}T\, d\tau = \int \vec{c}\,T(r)\cdot d\vec{a}$$

$$\vec{c}\cdot\int \vec{\nabla}T\, d\tau = \vec{c}\cdot\int T(r)\, d\vec{a}$$

Since true for all $\vec{c}$

$$\int \vec{\nabla}T\, d\tau = \int T(r)\, d\vec{a}$$

Consider $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ $\quad \vec{w} = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$

If $\vec{c}\cdot\vec{v} = \vec{c}\cdot\vec{w}$ for all $\vec{c}$ we can choose

$\vec{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ to prove $v_x = w_x$ and $\vec{c} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

to prove $v_y = w_y$ etc. This shows $\vec{v} = \vec{w}$!

7 cont'd

(b) $$\int_V \vec{\nabla}\cdot\vec{v}\, d\tau = \int_S \vec{v}\cdot d\vec{a}$$

plug in $\vec{v} \to \vec{v}\times\vec{c}$

and use $$\vec{\nabla}\cdot(\vec{v}\times\vec{c}) = \vec{c}\cdot(\vec{\nabla}\times\vec{v}) - \underbrace{\vec{v}(\vec{\nabla}\times\vec{c})}_{\phi \text{ since } \vec{c}=\text{const}}$$

$$\int_V \vec{c}\cdot(\vec{\nabla}\times\vec{v})\, d\tau = \int_S \underbrace{(\vec{v}\times\vec{c})\cdot d\vec{a}}_{\text{rewrite triple produc}}$$

$$\int_S \vec{c}\cdot(d\vec{a}\times\vec{v}) = -\int_S \vec{c}\cdot(\vec{v}\times d\vec{a})$$

$$\Rightarrow \quad \vec{c}\cdot\int_V(\vec{\nabla}\times\vec{v})\, d\tau = -\vec{c}\cdot\int_S(\vec{v}\times d\vec{a})$$

For any $\vec{c}$ $\Rightarrow$ $$\int(\vec{\nabla}\times\vec{v})\, d\tau = -\int_S \vec{v}\times d\vec{a}$$

(7 cont'd)

(c) $$\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{v}\,d\tau = \int_S \vec{v}\cdot d\vec{a}$$

Letting $\vec{v} = T\vec{\nabla}U$

use $\vec{\nabla}\cdot\vec{v} = \vec{\nabla}T\cdot\vec{\nabla}U + T\cdot\nabla^2 U$

and you are done

(d) Use (c) twice, once with $\vec{v} = T\vec{\nabla}U$

and once with $\vec{v} = U\vec{\nabla}T$

and subtract!

(e) $$\int_S \vec{\nabla}\times\vec{v}\cdot d\vec{a} = \int_P \vec{v}\cdot d\vec{l}$$

Let $\vec{v} = \vec{c}\,T$

$$\nabla\times(\vec{c}\,T) = T(\vec{\nabla}\times\vec{c}) - \vec{c}\times\vec{\nabla}T$$

$\phi$

$(\vec{c}\times\vec{\nabla}T)\cdot d\vec{a}$

$= -\vec{c}\cdot(\vec{\nabla}T\times d\vec{a})$

Triple product rewrite

$$-\int_S (\vec{c}\times\vec{\nabla}T)\cdot d\vec{a}$$

$$-\vec{c}\cdot\int_S \vec{\nabla}T\cdot d\vec{a} = c\cdot\int_P T\,d\vec{l}$$

$\boxed{8}$ $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2yz - 3y & x^3z - 3x & x^3y + 2z \end{vmatrix}$

conservative
$\Longleftrightarrow \vec{\nabla} \times \vec{F} = 0$
$\Longleftrightarrow \int_{\vec{a}}^{\vec{b}} \vec{F} \cdot d\vec{\ell}$
is path independent

$$= \hat{x}\left[x^3 - x^3\right] - \hat{y}\left[3x^2y - 3x^2y\right] + \hat{z}\left[3x^2z - 3 - 3x^2z + 3\right]$$

$$= \phi \quad \checkmark\checkmark$$

We know $\displaystyle\int_{\vec{a}}^{\vec{b}} \vec{\nabla}\phi \cdot d\vec{\ell} = \phi(\vec{b}) - \phi(\vec{a})$

So if $\vec{F} = -\vec{\nabla}\phi$

$$\int_{\vec{a}}^{\vec{b}} \vec{F} \cdot d\vec{\ell} = \phi(\vec{a}) - \phi(\vec{b})$$

choose $\vec{a} = (0,0,0)$ and $\phi(\vec{a}) = 0$ ← This is an arbitrary choice of the "constant of integration"

$\vec{b} = (x,y,z)$

Then $\displaystyle\phi(x,y,z) = -\int_{(000)}^{(xyz)} \vec{F} \cdot d\vec{\ell}$

Choose to get to $(x,y,z)$ by path along axes
(since answer is path independent, can pick any path...)

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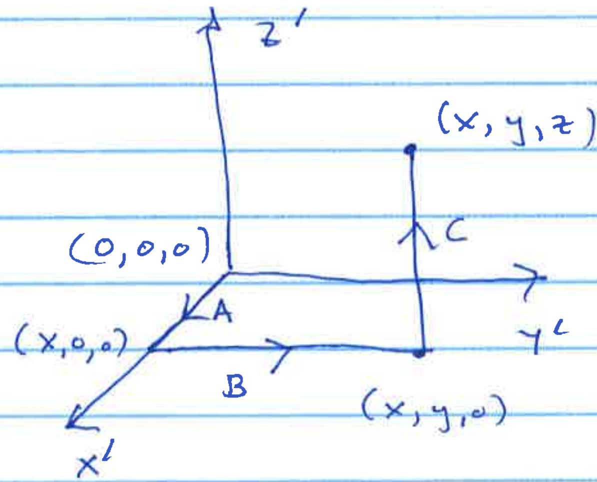
(8 cont'd) Choose path from (000) to (x, 0, 0)

to (x, y, 0) to (x, y, z):




along A:

$\vec{dl} = dx'\,\hat{x}'$ $\Rightarrow$ $\vec{F}\cdot\vec{dl} = F_x\,dx' = (3x'^2y'z' - 3y')\,dy' = \phi$

$y' = z' = 0$

along B:

$\vec{dl} = dy'\,\hat{y}'$ $\Rightarrow$ $\vec{F}\cdot\vec{dl} = F_y\,dy' = (x'^3 z' - 3x')\,dy'$

$x' = x$ $\qquad = -3x\,dy'$

$z' = 0$

contribution to $\phi$: $-\int_0^Y -3\,dy' = +3xY$

along C

$\vec{dl} = dz'\,\hat{z}'$ $\qquad \vec{F}\cdot\vec{dl} = (x'^3y' + 2z')\,dz'$

$x' = x$ $\qquad = (x^3y + 2z')\,dz'$

$Y' = Y$

contribution to $\phi$: $-\int_0^z (x^3y + 2z')\,dz' = -x^3yz - \frac{z^2}{2}$

(8 cont'd)

A + B + C $\quad \phi(x,y,z) = 3xy - x^3yz - z^2/2$

check $\quad -\frac{d\phi}{dx} = -3y + 3x^2yz$

$$-\frac{d\phi}{dy} = -3x + x^3 z$$

$$-\frac{d\phi}{dz} = x^3y + 2z$$

$$\Rightarrow \quad -\bar{\nabla}\phi = \vec{F} = (-3y + 3x^2yz)\hat{x} + (-3x + x^3z)\hat{y}$$

$$+ (x^3y + 2z)\hat{z} \qquad \checkmark\checkmark$$

$\boxed{9}$ First check $\vec{V}$ is solenoidal:

$$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \phi + \phi + \phi = \phi \quad \checkmark\checkmark$$

We want $\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \vec{V}$

So
$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = y + z$$

$$-\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z} = x - z$$

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = x^2 + y^2$$

Can use gauge freedom to choose any component of $\vec{A}$ to vanish. Let's select $A_x = 0$. Then

$$\frac{\partial A_y}{\partial x} = x^2 + y^2 \Rightarrow A_y = \frac{1}{3}x^3 + y^2 x + f(y,z)$$

$$\frac{\partial A_z}{\partial x} = z - x \Rightarrow A_z = zx - x^2/2 + g(y,z)$$

and
$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial g}{\partial y} - \frac{\partial f}{\partial z} = y + z$$

(9 cont'd) Can pick many choices for $f, g$.

One that works is $g = \frac{1}{2}y^2$ and $f = -z^2/2$

$$A_x = 0$$

$$A_y = \frac{1}{3}x^3 + y^2 x - z^2/2$$

$$A_z = zx - x^2/2 + \frac{1}{2}y^2$$

Then, to be sure, check

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = y + z \qquad = v_x \ \checkmark$$

$$\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0 - (z - x) = x - z \qquad = v_y \ \checkmark$$

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = x^2 + y^2 - 0 = x^2 + y^2 \qquad = v_z \ \checkmark$$

$\boxed{10}$ $$\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{v}\, d\tau = \int_{S} \vec{v}\cdot d\vec{a}$$

$\mathcal{V}: x^2+y^2+z^2 \le 25$ is sphere of radius $R = 5$ with center at origin

On surface $S$ of this sphere

$$\vec{v} = \underbrace{(x^2+y^2+z^2)}_{25}\underbrace{(x\hat{x}+y\hat{y}+z\hat{z})}_{\substack{\vec{r} = |\vec{r}|\hat{r} \\ = 5\hat{r}}} = 125\,\hat{r}$$

$$d\vec{a} = |da|\,\hat{r}$$

So $$\vec{v}\cdot d\vec{a} = 125\,|da|\,\underbrace{\hat{r}\cdot\hat{r}}_{1}$$

$$\int_S \vec{v}\cdot d\vec{a} = 125\underbrace{\int_S |d\vec{a}|}_{\substack{\text{surface} \\ \text{area of sphere}}} = 125\; 4\pi\underbrace{(25)}_{R^2} = 12500\pi$$

(10 cont'd) Can check by doing original $\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{v}\,d\tau$

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$= (3x^2+y^2+z^2) + (x^2+3y^2+z^2) + (x^2+y^2+3z^2)$$

$$= 5(x^2+y^2+z^2) = 5r^2$$

$$\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{v}\,d\tau = \int_0^R r^2 dr \underbrace{\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi}_{4\pi} \ \overbrace{5r^2}^{\vec{\nabla}\cdot\vec{v}}$$

$$= 4\pi\, 5 \int_0^{R=5} r^4\, dr$$

$$= 20\pi \left. r^5/5 \right|_0^5$$

$$= 20\pi\, 5^4 = 12500\,\pi$$

$\checkmark$